DEPARTMENT OF STATISTICS AND OPERATIONS RESEARCH
OPERATIONS RESEARCH DETERMINISTIC QUALIFYING EXAMINATION

August 10, 2015
9:00 am - 1 pm

General Instructions

This examination is closed-book, and consists of three questions. Answer all three as clearly and concisely as you are able. Use of the internet and/or mobile devices is not permitted.

Question 1. (20 points total)

Consider the Prize Collecting TSP: we have n cities, m arcs in the underlying directed graph, the travel cost from city i to city j is c_{ij} and there is a prize of value p_i at city i. The salesman wants to visit *some* cities, and return to the starting city 1. (So he must visit city 1.) We have that $c_{ij} \geq 0$ and $p_i > 0$ for all (i, j) and for all i.

If he visits a city i then he gets to pick up the prize at that city. He wants to maximize his profit, which is

- The sum of the prizes he collects, minus the total travel cost.

1. (10 points) Formulate this problem with $O(n + m)$ variables, and constraints.
2. (5 points) In addition to the above, add an MIP constraint to force:
   - If city 2 and city 5 are both visited, then city 2 must be visited before city 5.

   The total number of variables and constraints still should be $O(n + m)$.
3. (5 points) For this part, ignore part 2. Formulate the following using an MIP constraint. The total number of variables and constraints still should be $O(n + m)$.
   - If both city 2 and city 5 are visited, and city 5 is visited before city 2, then we must pay a penalty of C dollars, where C is a positive constant.

Question 2. (21 points total)

This question consists of three parts.

1. (7 points) Let $A \in \mathbb{R}^{m \times n}$ be given. Prove statements (a) and (b) below are equivalent.

   (a) There exists $x \in \mathbb{R}^n$ such that $Ax \geq 0$ and $Ax \neq 0$.

   (b) There does not exist $y \in \mathbb{R}^m$ such that $y^T A = 0$ and $y > 0$.

2. (7 points) Now consider statement (b') below:

   (b') There does not exist $y \in \mathbb{R}^m$ such that $y^T A = 0$, $y \geq 0$ and $y \neq 0$.

   Use an example to show that statements (a) and (b') are not equivalent.

3. (7 points) Fill in the blank below:

   (a') There exists $x \in \mathbb{R}^n$ such that ________________________________

   such that statement (a') is equivalent to statement (b'). Justify your answer.

Question 3. (10 points total)

Consider a polyhedron P defined as follows.

$$P = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} -1 & -6 & 1 \\ -1 & -2 & 7 \\ 0 & 3 & -10 \\ -6 & -11 & -2 \\ 1 & 6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} -3 \\ 4 \\ -7 \\ 0 \\ 6 \end{bmatrix} \right\}$$

For each of the following x^*, decide if x^* is an extreme point of the polyhedron P. If it is, find a vector $c \in \mathbb{R}^3$ such that x^* is the unique maximizer of $c^T x$ among all points $x \in P$. Otherwise, find two disjoint points x_1 and x_2 in P, so that x^* is the midpoint of x_1 and x_2.

1. $x^* = (1, 1, 1)$.
2. $x^* = (7, 0, 1)$.