OPERATIONS RESEARCH QUALIFYING EXAMINATION
DEPARTMENT OF STATISTICS AND OPERATIONS RESEARCH
—————————— AUGUST, 2016 ——————————

OR DETERMINISTIC QUALIFYING EXAMINATION

Information:

- **Student's full name:** ______________________________.
- **Student's signature:** ____________________.
- **Date:** August 15, 2016. **Time:** 9:00 AM - 1:00 PM.
- **Honor pledge:** I have neither given nor received unauthorized assistance on this exam.

General instructions: Please carefully read the following instructions:

a. Write your name on this sheet. Date and sign on the first page.

b. This examination is closed-book, and consists of three questions.

c. Answer all three as clearly and concisely as you are able.

d. Use of the internet and/or mobile devices is not permitted.

The exam questions

Question 1 (20 points): We have a directed graph G with $n = 50$ nodes, and $n(n-1) = 2450$ arcs, and arc costs denoted by c_{ij}. We have a distinguished source node s. The goal is to find the shortest path from s to all other nodes.

We first run the label correcting algorithm (LCA), and it finds that there is a negative cost directed cycle in the graph. With the computing resources that we have, we can do ALL of the following, but each of them only once: ?

(a) We can run the LCA on any graph with a million nodes, which can be assumed to be fully dense (i.e. every possible arc is in the graph), to check whether the graph has a negative cost directed cycle. If it does not, we can run the LCA to find shortest paths from a distinguished node.

(b) We can solve 1000 linear programming problems with at most 10,000 variables, and 15 thousand constraints each.

(c) We can run Dijkstra's algorithm on any graph with 10 million nodes, which can be assumed to be fully dense, as long as the graph has only nonnegative arc lengths.

(d) We can run the label setting algorithm on any graph with 10 million nodes, which can be assumed to be fully dense, as long as the graph is acyclic.

(e) We can run any algorithm that takes 10^{16} steps, such as comparison, addition, etc.

(f) We can solve 100 integer programming problems with at most 5000 variables, and 10 thousand constraints each.

(g) We can find a maximum flow in any graph with 10 million nodes, which can be assumed to be fully dense.

Explain how you would find the shortest paths in G from s to all other nodes.

If you want to do it using, say, option 1, then the total number of steps to construct the bigger graph with at most a million nodes, should be at most 10^{16}.

Question 2 (30 points): We consider the following convex optimization problem:

$$\begin{cases} \min\limits_{x\in\mathbb{R}^n} & c^T x \\ \text{s.t.} & \sup\limits_{a\in\mathcal{P}} a^T x \leq b. \end{cases} \tag{1}$$

where $\mathcal{P} := \{a \in \mathbb{R}^n : Ca \leq d\}$ is a given polyhedron such that $C \in \mathbb{R}^{m\times n}$ and $d \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ and $b \in \mathbb{R}$ are given. Here, we assume that $\mathcal{P}$ is nonempty.

(a) Clearly show that this optimization problem can be reformulated equivalently to a linear program (LP). Find its explicit formulation.

(b) Clearly derive the dual problem of this LP.

(c) Find the condition on the data C, c and d such that (1) is bounded.

Let us modify problem (1) as follows

$$\begin{cases} \min\limits_{x\in\mathbb{R}^n} & \max\limits_{c\in\mathcal{P}} c^T x \\ \text{s.t.} & Ax = b,\ \|x\|_\infty \leq \beta. \end{cases} \tag{2}$$

where $\beta > 0$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ are given, and $\mathcal{P}$ is a nonempty, closed, convex and bounded set in $\mathbb{R}^n$.

(d) If $\mathcal{P} := \{c : \|c\|_\infty \leq 1\}$, where $\|c\|_\infty = \max_{i=1,\cdots,n} |c_i|$, then show that (2) can be written equivalently to an LP. Find the explicit formulation of this LP.

(e) What is the relation between (2) and the following problem:

$$\begin{cases} \min\limits_{x\in\mathbb{R}^n} & \|x\|_1 \\ \text{s.t.} & Ax = b, \|x\|_\infty \leq \beta, \end{cases} \tag{3}$$

given that $\mathcal{P}$ is given as in Question (d), where $\|x\|_1 := \sum_{i=1}^n |x_i|$? Explain your answer clearly.

Question 3 (30 points): Let $a \in \mathbb{R}^n \setminus \{0\}$, $\beta \in \mathbb{R}$, and consider the hyperplane $H = \{x \in \mathbb{R}^n \mid a^T x = \beta\}$. For any $x_0 \in \mathbb{R}^n$, let

- $d(x_0, H)$ be the Euclidean distance from x_0 to H, and
- $h(x_0)$ be the point in H which is closest to x_0.

(a) Prove that $h(x_0)$ is unique.

(c) Give explicit formulas (in terms of a, β and x_0) for $d(x_0, H)$ and $h(x_0)$.

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ be given, and let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. Assume that P has a nonempty interior, i.e. that there is an x satisfying $Ax < b$. Let

- S be the largest sphere in $\mathbb{R}^n$ that is inscribed in P,
- c be the center of S,
- r be the radius of S.

(c) State a linear program which can be used to find S, c and r. Explain carefully how to use your LP to find these quantities. Can your LP also determine whether or not P has a nonempty interior? Explain.

——————— The end ———————