OPTIMIZATION QUALIFYING EXAMINATION
DEPARTMENT OF STATISTICS AND OPERATIONS RESEARCH
——————— AUGUST, 2018 ———————

OPTIMIZATION QUALIFYING EXAMINATION

Information:

- **Student's full name:** ____________________
- **Student's signature:** ____________________
- **Date:** August 13, 2018. **Time:** 9:00 AM - 1:00 PM.
- **Honor pledge:** I have neither given nor received unauthorized assistance on this exam.

General instructions:
Please carefully read the following instructions before answering.

a. Write your name on this sheet. Date and sign on the first page.

b. This examination is closed-books and notes, and consists of three questions.

c. Answer all three as clearly and concisely as you are able.

d. Use of the internet, computers, and/or mobile devices is not permitted.

The exam questions

Question 1: (35 points) We are given an alphabet $\{\alpha_1, \cdots, \alpha_L\}$ with L letters. Each α_i is called a letter. For example, in the English alphabet $L = 26$, $\alpha_1 = A$, $\alpha_2 = B$, $\ldots$, $\alpha_L = Z$.

We say that x is a *word* in this alphabet, if x is a finite sequence of letters. For example, $x = (\alpha_1, \alpha_5, \alpha_3)$ is a word.

If x and y are words, we say that y is a superword of x if y is obtained from x by inserting some other letters into x, but not changing the order of the original letters in x. For example, if $x = (\alpha_1, \alpha_5, \alpha_3)$ and $y = (\alpha_1, \underline{\alpha_7}, \underline{\alpha_5}, \alpha_5, \alpha_3)$, then y is a superword of x, where the inserted letters are underlined.

In the superword problem we are given

a. The alphabet as above;

b. Positive integers K and M;

c. Words $x^1, \ldots, x^K$, the length of x^i is n_i.

We are seeking a word y of length M such that it is a superword of each x^i. For example, if the alphabet is the English alphabet, $x^1 = SHU$, $x^2 = QUOC$, and $M = 6$, then a suitable y is $y = SHQUOC$.

Describe a 0–1 integer linear program (i.e., it should only have 0–1 variables), which is feasible, iff the y above exists. (Since only feasibility matters, the objective function does not matter.) The number of variables and constraints should be polynomial in L, M, and $N := \max_i n_i$. Carefully explain the meaning of all variables and constraints in detail.

Question 2: (30 points) Let $x^* = (2, 0, 1)$. Consider the following problem:

$$\begin{cases} \text{minimize} & 4x_1 + 2x_2 - x_3 \\ \text{subject to} & 2x_1 + x_2 - x_3 \geq 3 \\ & 6x_1 + 7x_2 - 5x_3 \geq 6 \\ & 6x_1 + x_2 - 3x_3 \geq 5 \\ & -x_1 - x_2 + x_3 \geq -1 \\ & x \geq 0 \end{cases} \tag{1}$$

a. Without using the simplex method, show that x^* is an optimal solution of (1).

b. Is the optimal solution of (1) unique? Justify.

c. Write down the dual problem and a dual optimal solution.

d. Let p_2 refer to the objective coefficient for x_2 whose current value is 2. What is the smallest value p_2 can take on such that x^* remains optimal?

Question 3: (35 points) Consider the following convex quadratic program with one linear equality constraint:

$$f(u) := \min_{x\in\mathbb{R}^n} \left\{ \frac{1}{2}x^T Qx + c^T x \ \text{ subject to } \ a^T x = u \right\}, \tag{2}$$

where Q is an $n \times n$-nonzero symmetric positive semidefinite matrix, $c \in \mathbb{R}^n$, $a \in \mathbb{R}^n$ such that $a \neq 0$, and $u \in \mathbb{R}$.

Part 1: First, assume that Q is just positive semidefinite (i.e., its smallest eigenvalue is zero). Find an example in $\mathbb{R}^n$ with $n \geq 2$ to illustrate the following three situations:

(i) Problem (2) has more than one optimal solution for any $u \in \mathbb{R}$.

(ii) Problem (2) has no optimal solution for any $u \in \mathbb{R}$.

(iii) Problem (2) has a unique optimal solution for any $u \in \mathbb{R}$.

Part 2: Now, we assume that Q is positive definite (i.e., its smallest eigenvalue is positive). Here, Q, c and a are fixed. Solve the following questions:

a. Show that problem (2) always has a unique optimal solution $x^*(u)$ for any $u \in \mathbb{R}$. Explicitly compute the optimal solution $x^*(u)$, and its derivative with respect to u.

b. Show that the optimal value function $f(u)$ defined in (2) is convex with respect to $u \in \mathbb{R}$.

c. Given $f(u)$ computed in Question (b), show that this function is also a quadratic function. Then, solve the following problem

$$\min_{u\in\mathbb{R}} f(u),$$

and compute its optimal value.

The end