DEPARTMENT OF STATISTICS AND OPERATIONS RESEARCH
THE UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL
————————— AUGUST, 2019 —————————

OPTIMIZATION QUALIFYING EXAMINATION

Information:

- **Student's full name:** ________________________________
- **Student's signature:** ____________________
- **Date:** August 13, 2019. **Time:** 9:00 AM - 1:00 PM.
- **Honor pledge:** I have neither given nor received unauthorized assistance on this exam.

General instructions: Please carefully read the following instructions before answering.

a. Write your name on this sheet. Date and sign on the first page.
b. This examination is closed-books and notes, and consists of three questions.
c. Answer all three as clearly and concisely as you are able.
d. Use of the internet, computers, and/or mobile devices is not permitted.

The exam questions

Question 1: (30 points) We are given a matrix A a vector b and ℓ row vectors $d_1, \dots, d_\ell$. We know that $d_i x > 0$ for $i = 1, \dots, \ell$ whenever $Ax \leq b$, where x is the vector of decision variables.

Formulate the problem

$$\begin{cases} \max_x & \prod_{i=1}^{\ell} (d_i x) \\ \text{s.t.} & Ax \leq b \end{cases} \tag{1}$$

as an SOCP (second-order cone program). You can use $\mathcal{O}(\ell)$ extra variables and constraints.

a. Do this rigorously, with a proof of correctness when $\ell = 8$ and when $\ell = 5$. (Hint: first try $\ell = 2$, and $\ell = 4$ to get some intuition).

b. Outline how you would do the formulation for general $\ell \geq 2$.

Question 2: (40 points) Consider the following linear program:

$$\begin{cases} \min_{x,z} & c^T x + d^T z \\ \text{s.t.} & Ax \quad + Bz \geq -c \\ & -B^T x + Cz = -d \\ & x \geq 0, \ z \text{ free}, \end{cases} \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, and $d \in \mathbb{R}^m$.

(a) Write down the dual problem of (2). Find the conditions on A and C such that the primal problem (2) is the same as the dual problem (i.e. the primal linear program (2) and its dual form are equivalent).

(b) One consequence of the weak duality theorem in LP states that: "if the primal linear program is unbounded then the corresponding dual problem is infeasible". Prove that the following statement: "If the primal linear program is infeasible, then the corresponding dual problem is unbounded" fails to hold by constructing a counterexample of the form (2).

(c) Under the conditions found in (a), use the strong duality theorem to find a necessary and sufficient condition for a feasible solution (x, z) of (2) to be optimal. Prove your claim mathematically.

(d) We consider a linear programming instance of (2) with the following data (A, B, C, c, d):

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad \text{and } d = \begin{bmatrix} 4 \\ 5 \end{bmatrix}. \tag{3}$$

Given $x^\star = (\frac{3}{4}, 0, \frac{1}{4})^T$, find $z^\star$ such that $(x^\star, z^\star)$ is a feasible solution of (2). Without using complementarity slackness, prove that $(x^\star, z^\star)$ is an optimal solution of (2). Is this solution unique to the given linear programming instance? Justify your answer.

(e) Given (2) with the input data in (3), assume that we perturb the component $c_1 = -1$ in (3) to $c_1 = -1 + \delta$. Find the range of δ such that the optimal solution found in (d) remains optimal to the perturbed linear programming instance.

Question 3: (30 points) Consider the following optimization problem:

$$\min_{\beta} \|\beta - y\|_2^2 + \lambda|\beta|_1 \tag{4}$$

where $\beta \in \mathbb{R}^n$ is the variable, $y \in \mathbb{R}^n$ and $\lambda > 0$ are parameters. For a vector $\beta \in \mathbb{R}^n$, $\|\beta\|_2^2 = \sum_{i=1}^n \beta_i^2$ is its squared Euclidean norm (i.e., the sum of squares of its components), and $|\beta|_1 = \sum_{i=1}^n |\beta_i|$ is its ℓ_1-norm (i.e., the sum of absolute values of its components).

a. Convert (4) into a quadratic program by introducing more variables if necessary.

b. Write down the KKT (Karush-Kuhn-Tucker) conditions for the quadratic program obtained in Part a.

c. Give a closed-form formula for the optimal solution of (4) in terms of y and λ. Justify your solution mathematically.

d. Suppose that β^* is the optimal solution for (4). Let $s = \|\beta^*\|_1$. Prove that β^* is also the optimal solution to the following optimization problem, in which y and s are parameters:

$$\begin{cases} \min_{\beta} \ \|\beta - y\|_2^2 \\ \text{s.t. } \|\beta\|_1 \leq s. \end{cases} \tag{5}$$

The end