## STOR641 - Comprehensive Written Exam - August 2015

**This test consists of three questions. The relative weights are specified following each part. You are NOT allowed to refer to any notes, books etc.**

### Good luck!

**1−** At the end of every hour, a single job arrives at a single-server system with probability $\alpha > 0$ independently of everything else. The total system capacity is 1 meaning that newly arriving jobs are sent away whenever there is already a job in the system (receiving service or waiting for the server to come back from a break). Jobs can be either easy or difficult. Each job is difficult with probability $p$ $(0 < p < 1)$ independently of everything else. If the job is easy, it takes the server 50 minutes to serve the job; if the job is difficult, it takes the server 2 hours. At the beginning of every hour, unless the server is in the middle of serving a difficult job, it checks to see if there is a job waiting. If there is no job, the server simply waits idly for an hour and checks again to see whether there is a job. Otherwise, the server picks the job and serves it. After serving an easy job, the server is immediately available to pick another job at the top of the hour. After serving a difficult job, however, the server needs to take a break during which time the server does not serve any job. After coming back from the break, the server waits idly until the beginning of the next hour and then checks again to see if there are any jobs and continues with the service process as described above. The break times are independent and identically distributed with an exponential distribution with parameter $\lambda > 0$ per hour.

(a) Model this system as a discrete-time Markov chain (DTMC). Clearly describe the states and give the transition probabilities. **(10 points)**

(b) Write the balance equations. DO NOT SOLVE THEM. **(5 points)**

(c) Suppose that a customer who has just arrived found the server on a break. What is the expected time until this job starts receiving service? **(5 points)**

(d) Letting $\pi_i$ denote the steady-state probability that the system is in state $i$, give an expression for the fraction of jobs which are denied access to the system because of capacity. You are NOT expected to solve the balance equations. **(10 points)**

(e) Letting $\pi_i$ denote the steady-state probability that the system is in state $i$, give an expression for the long-run average rate with which jobs (easy or difficult) leave the system after receiving service. You are NOT expected to solve the balance equations. **(10 points)**

(f) Letting $\pi_i$ denote the steady-state probability that the system is in state $i$, give an expression for the long-run fraction of time the server is busy serving a job. You are NOT expected to solve the balance equations. **(10 points)**

**2−** Let $\{X(t), t \geq 0\}$ be a continuous-time Markov chain (CTMC) on $\{0, 1, 2, \dots\}$ with transition rates given by $q_{0i} = q_{i0} = \lambda_i > 0$ for $i \geq 1$ with $\sum_{j=1}^{\infty} \lambda_j = \lambda < \infty$. All other transition rates are zero. Is this CTMC irreducible? Is it recurrent? Is it positive recurrent? Explain and show why or why not. **(30 points)**

**3−** Consider a single-server service system in which jobs are served in a last-come-first-served fashion. However, service is non-preemptive meaning that the server always finishes the service of a job it has started serving before moving onto a new job even if a new job arrives in the mean time.

The system capacity is three meaning that the total number of customers in the system including the customer in service at any given time cannot exceed three. Arrivals occur according to an independent Poisson process with rate $\lambda > 0$ and service times are independent and identically distributed with an exponential distribution with mean $1/\mu$ where $0 < \mu < \infty$. Suppose that an arriving customer found the server busy with serving a customer but no other customer waiting in the queue. What is the expected time this customer spends in the system including the time spent in service? **(20 points)**