

STOR641 - Comprehensive Written Exam - August 2016

This test consists of three questions. The relative weights are specified following each part. You are NOT allowed to refer to any notes, books etc.

Good luck!

1– For each one of the following two statements, first state whether or not the statement is true or false, then provide a rigorous explanation or proof in support of your response.

- (a) Any finite state discrete-time Markov chain has a limiting distribution. (15 points)
- (b) If a regular continuous-time Markov chain is positive recurrent and has a limiting distribution, then its embedded discrete-time Markov chain is also positive recurrent and has a limiting distribution. (15 points)

2– Consider a service system in which there are three identical servers serving customers in a first-come-first-served fashion. Service time for each customer is independent and exponentially distributed with mean  $1/\mu$ . Customers who wait in the queue are possibly impatient and they leave the queue without receiving service if their queueing time exceeds their patience time. The patience time for each impatient customer is independent (of each other as well as the queue length and the service process) and exponentially distributed with mean  $1/\theta$ . All customers, when in service, are patient and wait until their service is complete.

- (a) Suppose that a patient customer, who is guaranteed to wait until receiving service, has just arrived. All three servers are busy and there is only one other customer waiting and this customer is impatient. What is the expected time until the customer who has just arrived leaves the system. (15 points)
- (b) Suppose now that the customer who has just arrived is also impatient along with the customer who is already waiting in the queue. What is the probability that the customer who has just arrived will receive service? (15 points)

3– Consider a theme park which is accessible to its visitors only through a monorail system. Specifically, all visitors must arrive at Station A, and there, they must take the train to Station B, where the theme park is located. (Note that there is a single train that operates between Station A and Station B.) The train has a maximum capacity of  $K$  passengers. Visitors arrive at Station A according to an independent Poisson process with rate  $\lambda$  visitors per minute. Each arriving visitor joins the queue for the next available train. Once at Station B, each visitor on the train immediately gets off the train, spends some time at the theme park and comes back at Station B to take the train back to Station A. The time each visitor spends at the theme park is independent and exponentially distributed with mean  $1/\mu$  minutes. The time it takes the visitors to get from Station B to the theme park can be assumed to be negligible. Just like at Station A, every visitor arriving at Station B joins the queue for the next available train. It takes the train 10 minutes to get from Station A to Station B or from Station B to Station A. When the train arrives at one of the stations, all the passengers on the train get off and new passengers get on the train in a first-come-first-served fashion upto the train's capacity of  $K$  passengers. Passengers who cannot get on the train continue waiting in the queue until the next available train. We assume that passengers get on and off the train instantaneously so that the train does not spend any time in any of the two stations and leaves the station with new passengers as soon as it arrives.

- (a) Model this system as a discrete-time Markov chain (DTMC). Clearly describe the state space and give the transition probabilities. DO NOT GIVE THE BALANCE EQUATIONS. (Hint: One possibility is to model the system so that the system state is described as a four-dimensional vector with entries corresponding to the number of visitors waiting at Station A, number of visitors waiting at Station B, number of visitors in the theme park, and the location of the train right after (or before) the train arrives at a station.) (20 points)
- (b) Suppose that the DTMC you described in part (a) is positive recurrent and let  $\pi(i)$  denote the steady-state probability that the system is in state  $i$ . Also let  $M$  denote the number of passengers on the train that leaves station B in the steady state. Give an expression for  $P\{M = m\}$  for  $m = 0, 1, \dots, K$ . Give your answer in terms of  $\pi(i)$ s. (10 points)
- (c) Suppose that the DTMC you described in part (a) is positive recurrent and let  $\pi(i)$  denote the steady-state probability that the system is in state  $i$ . What fraction of the train trips (either from Station A to Station B or from Station B to Station A) are done at full capacity with  $K$  passengers? Give your answer in terms of  $\pi(i)$ s. (10 points)