STOR641 - Comprehensive Written Exam - August 2018

This test consists of three questions. The relative weights are specified following each part. You are NOT allowed to refer to any notes, books etc.

Good luck!

1− Consider two single-server queues. Arrivals at each queue follow independent Poisson processes with rates λ_1 and λ_2, respectively. Service times in queue i (for $i = 1, 2$) are independent and identically distributed with an exponential distribution with parameter μ_i. As soon as server i (for $i = 1, 2$) becomes idle (which happens only when there are no customers in queue i), it sends out a signal to each customer in queue $3 - i$ indicating that it is free to serve a customer. Upon receiving this signal, each customer waiting (but not being served) in queue $3 - i$ leaves its queue position and starts moving towards queue i. The time it takes each customer to go from queue $3 - i$ to queue i is independent and identically distributed with an exponential distribution with parameter θ_{3-i}. The server starts working again as soon as a customer arrives (either from the other queue or from outside the system). When the server starts working again, it does not send another signal and so customers who are already in transit from one queue to the other continue their journey. Note also that when a server becomes idle, it sends a signal only to the customers waiting in the other queue, not to any of the customers who are traveling from one queue to the other.

- **(a)** Model this system as a continuous-time Markov chain (CTMC). Clearly describe the state space $\mathcal{S}$ and give the transition rates.**(20 points)**
- **(b)** Suppose that this CTMC is positive recurrent and for any $j \in \mathcal{S}$, let p_j denote the steady-state probability that the system is in state j. Write down the balance equation only for the state that represents the empty system. **(5 points)**
- **(c)** In the long-run, what fraction of the customers are served by Server 1? Do NOT solve the balance equations. Simply give your answer using p_js. **(10 points)**
- **(d)** In the steady-state, what fraction of the customers in the system are traveling from one queue to the other. Do NOT solve the balance equations. Simply give your answer using p_js. **(10 points)**

2− Consider a homogeneous Poisson process with rate $\lambda > 0$. Assume that $t > 2a$ for some fixed positive values of t and a and determine the following:

- **(a)** What is the probability that the time between the first and second event is more than a?**(5 points)**
- **(b)** If exactly one event has occurred by time t, what is the probability that the time between the first and second event is more than a?**(10 points)**
- **(c)** If exactly two events have occurred by time t, what is the probability that the time between the first and second event is more than a?**(10 points)**

3− Travis likes to play a game in which if he has $i > 0$ dollars and wins he gets another i dollars and his fortune grows to $2i$ dollars. If he loses the game, he has to give 1 dollar and so his fortune goes down to $i - 1$. The probability of his winning the game whenever he has i dollars independently of everything else is p/i and losing the game is $1 - p/i$ where $0 < p < 1$. Suppose that Travis plays this game repeatedly and as soon as his fortune goes down to zero, his father gives him one dollar so that he can continue to play the game. Let X_n denote Travis' fortune at the end of game n.

(a) Model $\{X_n, n \geq 0\}$ as a discrete time Markov chain clearly describing the state space and providing the transition probabilities.**(5 points)**

(b) Is $\{X_n, n \geq 0\}$ irreducible?**(5 points)**

(c) Suppose that $p = 0.4$. Is $\{X_n, n \geq 0\}$ positive recurrent?**(10 points)**

(d) Now, suppose that Travis starts with one dollar and stops playing the game either when he ends up having four dollars or zero dollars. (So, his father does not give him another dollar if he loses his money.) What is the probability that he goes broke at the end? (Assume again that $p = 0.4$.)**(10 points)**