STOR 642
Comprehensive Written Examination
9:00am-1:00pm, August 18, 2016

This test consists of three questions.
This is a closed book exam. No communication devices of any kind are allowed.
Explain your answers in detail.
The problem and subproblem weights are given in parentheses.

Problem 1. (20 points) New patients arrive according to a PP(λ) to a health care clinic staffed with one nurse (Matt) and two doctors (Drs. Davis and Sharma). The new patients first see Matt who does a preliminary evaluation. He routes the patient to Dr. Davis with probability α and to Dr. Sharma with probability $1-\alpha$. The preliminary evaluations take iid exp(μ_0) times. Dr. Davis takes an exp(μ_1) time to treat his patients, while Dr. Sharma takes an exp(μ_2) amount of time to treat her patients. All treatment times are independent. After seeing Dr. Davis a patient leaves with probability θ_1, or, with probability $1-\theta_1$, is asked to return to the clinic after a random amount of time with mean τ_1 to see him again. After seeing Dr. Sharma a patient leaves with probability θ_2, or, with probability $1-\theta_2$, is asked to return to the clinic after a random amount of time with mean τ_2 to see her again. The returning patients do not see Matt, but go straight to their assigned doctors.

a. (4) Make appropriate assumptions to model the above clinic as a Jackson network. What are its parameters?

b. (4) Write the traffic equation and solve it.

c. (4) When is the network stable?

d. (4) Assuming stability, compute the expected number of customers (including those in service) (*i*) waiting in the clinic, (*ii*) waiting for Dr. Davis and (*iii*) waiting for Dr. Sharma.

e. (4) What is the expected time a new patient spends in the clinic until he finally leaves the clinic permanently? (Hint: Apply Little's Law to the whole network.)

Problem 2. (15 points)

a. (5) Define an alternating renewal process $\{Z(t), t \geq 0\}$ with state-space $\{0, 1\}$.

b. (5) Assume the alternating renewal process has entered state 0 at time 0. Derive a renewal type equation for

$$H(t) = P(Z(t) = 0).$$

c. (5) When does $H(t)$ have a limit as $t \to \infty$? Use Key renewal theorem to compute this limit when it exists.

Problem 3. (15 points) Let $\{B(t), t \geq 0\}$ be a standard Brownian motion. Define, for $x \in [-1, 1]$,

$$T(x) = \min\{t \geq 0 : (x + B(t))^2 = 1\}.$$

Let $m(x) = E(T(x))$.

a. (5) Show that $m(x)$, $x \in [-1, 1]$, satisfies the following differential equation

$$m''(x) = -2.$$

State the boundary conditions.

b. (5) Solve the equation. Compute $m(0)$.

c. (5) Show that $\{B^2(t) - t, t \geq 0\}$ is a Martingale with respect to $\{B(t), t \geq 0\}$. Compute $m(0)$ by applying the optional sampling theorem to this martingale.