STOR 642
Comprehensive Written Examination
9:00am-1:00pm, August 17, 2017

This test consists of four questions.
This is a closed book exam. No communication devices of any kind are allowed.
Explain your answers in detail.
The problem and subproblem weights are given in parentheses.

Problem 1. (20 points) Consider an $M|G|1$ queue with arrival rate λ, and service times with distribution $G(\cdot)$, mean τ and second moment s^2. Suppose the first service starts on a new customer at time 0. Let A_n be the the number of arrivals during the nth service time, and X_n be the number of customers in the system after the nth service completes.

a. (5) Are $\{A_n, n \geq 0\}$ iid? Why or why not? Compute $E(A_n)$ and $E(A_n^2)$.

b. (5) Express X_{n+1} in terms of X_n and A_{n+1}.

c. (5) Take expectations on both sides of the equation derived in part b above. Assuming the limits exist, compute $\lim_{n\to\infty} P(X_n = 0)$.

d. (5) Now square both sides of the equation in part b, and then take expectation on both sides. Assuming the limits exists, compute $\lim_{n\to\infty} E(X_n)$.

Problem 2. (15 points)

a. (5) Define a renewal reward process $\{Z(t), t \geq 0\}$.

b. (10) Using the almost sure version of elementary renewal theorem, show that

$$\lim_{t\to\infty} \frac{Z(t)}{t} = \frac{r}{\tau},$$

with probability one, where τ is the mean cycle time and r is the mean reward in each cycle.

Problem 3. (10 points) Let $\{B(t), t \geq 0\}$ be a standard Brownian motion. Define, for $x \in [-1, 1]$,

$$T(x) = \min\{t \geq 0 : B(t) + x = \pm 1\}.$$

Let $c(x) = E(\int_0^{T(x)} (B(u) + x)^2 du)$.

a. (5) Derive a differential equation satisfied by $c(x)$, $x \in [-1, 1]$. State the boundary conditions.

b. (5) Solve the equation. Compute $c(0)$.

Problem 4. (5 points) Let $\{B(t), t \geq 0\}$ be a standard Brownian motion. Show that

$$Y(t) = \exp(\theta B(t) - \theta^2 t/2)$$

is a Martingale for any real valued θ.