STOR 642
Comprehensive Written Examination
9:00am-1:00pm, August 15, 2019

This test consists of two questions.
This is a closed book exam. No communication devices of any kind are allowed.
Explain your answers in detail.
The problem and subproblem weights are given in parentheses.

Problem 1. (50 points) Consider n single server queues in tandem. The service times at the kth queue are iid Exp(μ) and it has no waiting room (that is, capacity is one) ($1 \leq k \leq n$). Customers arrive from outside to the first queue according to a Poisson process with rate λ. If the first server is busy customer tries the second server, and so on. Thus an arriving customer gets into service at the first idle server. If all serves are busy he leaves the system without receiving service. Let $X_k(t)$ be the number of customers at service station k, and $Z_k(t) = X_1(t) + \cdots + X_k(t)$, ($1 \leq k \leq n$).

1. (10) Show that the first service station can be modeled as an M/M/1/1 queue.
2. (10) Show that $\{Z_k(t), t \geq 0\}$ is the queue length process in an $M/M/k/k$ queue. Compute the limiting distribution of $Z_k(t)$.
3. (10) Show that kth service station can be modeled as a G/M/1/1 queue, $1 \leq k \leq n$.
4. (10) Compute the steady state rate of arrivals to service station k. Explicitly state if and where you used PASTA.
5. (10) Compute the steady state expected inter-departure times of customers who leave without service. Explicitly state if and where you used elementary renewal theorem.

Problem 2. (50 points) A dealer buys and sells one item (say used cars) at times n=0,1,2,... At each time n he has an opportunity to buy one item and/or sell one item (he can do both). He controls the inventory by setting the buying price (called the bid) and the selling price (called the ask). Both the ask and bid prices belong to the finite set $D = \{p_1, p_2, \cdots, p_k\}$. Assume p_i increases in i. If the bid price is $b \in D$, the buy occurs with probability $\beta(b)$. If the ask price is $a \in D$, the sell occurs with probability $\alpha(a)$). We assume that $\beta(b)$ is an increasing function of b with $\beta(p_1) = 0, \beta(p_k) = 1$, and $\alpha(a)$ is a decreasing function of a with $\alpha(p_1) = 1, \alpha(p_k) = 0$. Every time the dealer buys an item at price b his cash reserve goes down by b, and every time he sells an item at price a, his cash reserve goes up by a. He wants to maximize the expected total discounted net revenue (discount factor $\delta \in [0, 1)$). Let X_n be the inventory at time n, before the transactions in period n are completed. Assume that he can't sell an item if $X_n = 0$, and can't buy an item if $X_n = B$, where $B > 0$ is the upper bound on the inventory size. Let $v(i)$ be the optimal expected total discounted net revenue over the infinite horizon starting with the initial inventory i.

1. (25) Formulate this as a Markov Decision Process by identifying the state space, action space, the transition probabilities, and the cost functions.
2. (5) State the optimality equation satisfied by the optimal value function v.
3. (5) Show how you can obtain the optimal ask and bid prices in state i if you know the function v.
4. (5) Show how the optimality equation can be solved by value iteration. State the termination criterion.
5. (5) Show how the optimality equation can be solved by policy iteration. State the termination criterion.
6. (5) Show how the optimality equation can be solved by linear programming.