August 2015 Name:_____

Comprehensive written exam – STOR654 Mathematical Statistics

All problem parts have equal weight. In budgeting your time expect that some part will take longer than others. When solving multi-part problems feel free to use results of earlier parts even if you cannot solve them in proving later parts. Do not forget to split the time between both papers.

1. Is it possible for a random vector (X, Y) to have the following properties: $EX = 6, EY = 5, EX^2 = 42, EY^2 = 29, EXY = 25$? (If yes, show an example, if no explain.)

2. Let $X_1, \ldots, X_n$ be i.i.d. with density $f(x|\mu, \lambda) = \lambda e^{-\lambda(x-\mu)} I_{(\mu,\infty)}(x)$.

   (a) Find the moment generating function of $X_{(k)}$

   (b) Find the joint distribution of $(X_{(1)}, X_{(2)} - X_{(1)})$. Are they independent?

   (c) Find MM estimator of μ and λ.

   (d) Find ML estimator of μ and λ.

   (e) Is the MLE of $1/\lambda$ unbiased?

3. Let $\pi(\theta|x)$ be a posterior distribution based on data x.

   (a) The predictive distribution of $Y|x$ is given by $\int f(y|x, \theta')\pi(\theta'|x)\, d\theta'$. Let $X \sim N(\theta, n^{-1})$ and $Y \sim N(\theta, m^{-1})$ be independent. Find the predictive distribution $Y|x$ using an improper prior $\pi(\theta) = 1$. Also find the posterior $\pi(\theta|x, y)$.

   (b) Let us generate y from the predictive distribution and consider the expected posterior based on both the observed and generated data $\pi(\theta|x, y)$:

   $$\tilde{\pi}(\theta|x) = \int \pi(\theta|x, y) \int f(y|x, \theta')\pi(\theta'|x)\, d\theta'\, dy.$$

   What is the relationship between $\tilde{\pi}(\theta|x)$ and $\pi(\theta|x)$? (Hint: This part does not assume normal distribution!)

4. Let X be a random variable with **distribution** function

$$F_X(x|\lambda) = e^{-e^{-\lambda x}}, \qquad \lambda > 0.$$

(a) Is this an exponential family?

(b) Find a 95% confidence interval. Evaluate the CI numerically for $x = -0.3$.

(c) Find the uniformly most powerful 0.05-level test for testing the $\mathcal{H}_0 : \lambda = 1$ vs. $\mathcal{H}_1 : \lambda < 1$. Evaluate its p-value for $x = -0.3$.