STOR 762
Comprehensive Written Exam
August 12, 2015

- This exam consists of 3 questions on 2 pages.
- The exam is closed book and closed notes.
- You are NOT allowed to use a calculator or a cell phone throughout the exam.
- Explain your answers in detail.

Problem 1. (30 points) Let X be an Erlang-2 random variable with probability density function $f(x) = xe^{-x}$, $x \geq 0$.

(a) Provide an acceptance-rejection algorithm to generate random variates from $f(x)$ using the majorizing function $t(x) = 2e^{-1-x/2}$, $x \geq 0$.

(b) Provide a convolution algorithm to generate random variates from $f(x)$.

(c) Compare the algorithms provided in parts **(a)** and **(b)** in terms of computational efficiency.

Problem 2. (40 points) Consider processes $\{X_i\}_{i\geq 1}$ and $\{Y_i\}_{i\geq 1}$ defined by

$$X_i = X_{i-1} + \epsilon_i, \text{ for } i = 1, 2, \ldots,$$

and

$$Y_i = Y_{i-1} + \theta_i, \text{ for } i = 1, 2, \ldots,$$

where $X_0 = Y_0 = 1$, $\{\epsilon_i\}_{i\geq 1}$ is a sequence of independent and uniformly distributed random variables over $[-a, a]$ for real $a > 0$, and $\{\theta_i\}_{i\geq 1}$ is another sequence of independent and uniformly distributed random variables over $[-b, b]$ for real $b > 0$. To estimate the difference in the long-run average mean of these two processes, a single simulation run of length n observations for each process is conducted by using common random numbers (CRN). Answer the following questions assuming that the inverse-transform method is used for the generation of all random variates.

(a) Obtain an expression for Y_i in terms of X_i for $i = 1, 2, \ldots, n$ under CRN.

(b) Let $\bar{X}_n$ and $\bar{Y}_n$ denote the sample mean from the single run conducted for processes $\{X_i\}_{i\geq 1}$ and $\{Y_i\}_{i\geq 1}$, respectively. Show that $Var(\bar{Y}_n) = (b/a)^2 Var(\bar{X}_n)$ and $Cov(\bar{X}_n, \bar{Y}_n) = (b/a)Var(\bar{X}_n)$ under CRN.

(c) Obtain an expression for $Var(\bar{X}_n - \bar{Y}_n)$ in terms of $Var(\bar{X}_n)$ under CRN.

(d) Suppose that instead of using CRN, an independent stream of random numbers are used to generate $\{Y_i\}_{i\geq 1}$. Obtain an expression for $Var(\bar{X}_n - \bar{Y}_n)$ in terms of $Var(\bar{X}_n)$ under this independent sampling procedure.

(e) Based on your answers to parts **(c)** and **(d)**, what is the percent reduction in $Var(\bar{X}_n - \bar{Y}_n)$ achieved by using common random numbers instead of independent sampling if $b = 2a$?

(f) Explain how you would construct a 95% confidence interval on the long-run average difference of means of these two processes using the method of batch means with b batches and the CRN method. Be specific about the upper and lower bounds of the confidence interval that you obtain (i.e., express them as functions of $\{X_i, i = 1, 2, \ldots, n\}$ and $\{Y_i, i = 1, 2, \ldots, n\}$).

(g) Discuss what your answer to part **(e)** suggests for the quality of the confidence interval given in part **(f)**.

Problem 3. (30 points) Let $X_1, X_2, \ldots, X_n$ be a finite sequence of independent and identically distributed observations with mean $\mu < \infty$ and variance $\sigma^2 < \infty$. Answer the following questions regarding the mean estimator $\hat{\mu} = \sum_{i=1}^{n} a_i X_i$, where a_i's are some real numbers.

(a) Under what condition(s) on $\{a_i\}_{i=1}^{n}$, $\hat{\mu}$ is an unbiased estimator of μ?

(b) Provide a sequence $\{a_i\}_{i=1}^{n}$ that satisfies the condition(s) given in part **(a)** and also yields the smallest variance for the mean estimator $\hat{\mu}$.

(c) Discuss whether the mean estimator identified in part **(b)** is a consistent estimator for μ.