STOR 762
Comprehensive Written Exam
August 16, 2017

- This exam consists of 3 questions on 2 pages.
- The exam is closed book and closed notes.
- You are NOT allowed to use a calculator or a cell phone during the exam.
- Explain your answers in detail.
- You may need the following information:
  - If $X \sim$Uniform(a, b), then $\mathrm{E}[X] = (a + b)/2$ and $\mathrm{Var}(X) = (b - a)^2/12$.

Problem 1. (40 points) Consider the discrete-time Markov chain $\{X_i, i \geq 1\}$ with state space $\{0, 1\}$, where

$$X_{i+1} = \begin{cases} X_i & \text{with probability } \alpha \\ 1 - X_i & \text{with probability } 1 - \alpha, \end{cases}$$

for $i \geq 1$, $\alpha \in [1/2, 1)$, and $X_0 = 0$.

(a) Show that $\mathrm{E}[X_i] = (1 - (2\alpha - 1)^i)/2$ for $i = 1, 2, \ldots$ Hint: Condition on X_{i-1}.

(b) Suppose that k independent replications of this Markov chain (each with $\lfloor n/k \rfloor$ observations) are conducted without any truncation. Provide a point estimator for its steady-state mean based on the observations collected.

(c) It is in fact easy to obtain the steady-state mean of this Markov chain analytically as $1/2$. Obtain the absolute bias for the point estimator provided in part **(b)**. (Absolute bias of an estimator is the absolute value of the difference between the expected value of the estimator and the actual value of the parameter to be estimated.)

(d) If everything else is fixed, show that the absolute bias obtained in part **(c)** is minimized at $k = 1$. Explain why this makes intuitive sense.

Problem 2. (25 points) Discrete-event simulation is used to evaluate four possible designs for a manufacturing system. Let μ_i be the throughput of system configuration $i \in \{1, 2, 3, 4\}$. For comparison purposes, two-sided confidence intervals on all pairwise differences of throughput of these system configurations are constructed with an overall confidence level of 95%. Suppose that n replications for each system using common random numbers are completed and the following data is obtained: $\bar{X}_{i,j}$ for $i \in \{1, 2, 3, 4\}$ and $j = 1, 2, \ldots, n$, where $\bar{X}_{i,j}$ is the average throughput of system i from replication j.

(a) Explain how you would obtain the confidence intervals. Be specific about the upper and lower bounds of the confidence intervals that you obtain (i.e., express them as functions of $\bar{X}_{i,j}$ and n).

(b) If all the confidence intervals obtained above include zero, what conclusions can be drawn about these four designs and what would be the next steps?

Problem 3. (35 points) Consider a first-order moving average process $\{X_i\}_{i \geq 1}$, which is is defined by

$$X_i = \phi \epsilon_{i-1} + \epsilon_i, \text{ for } i = 1, 2, \ldots,$$

where $|\phi| < 1$ is a constant and ϵ_i's are independent and uniformly distributed over the interval (a, b) with $a < b$. A simulation experiment is used to construct a confidence interval on the steady-state mean of this process. For this purpose, 20 replications each of length n observations are conducted and the inverse-transform method is used for the generation of all uniform variates. Let $X_i^{(j)}$ be the ith observation from replication j. Let also $\{\epsilon_i^{(j)}\}_{i=0}^{n}$ be the sequence of uniform variates used in replication j.

(a) Find an expression for $\epsilon_i^{(2k)}$ in terms of $\epsilon_i^{(2k-1)}$ for $i = 0, 1, \ldots, n$ and $k = 1, \ldots, 10$ assuming that replication $2k$ uses the antithetic variates from replication $2k - 1$.

(b) Obtain the expected value and variance of $(\epsilon_i^{(2k-1)} + \epsilon_i^{(2k)})/2$ for $i = 0, 1, \ldots, n$ and $k = 1, \ldots, 10$ under both antithetic variates and independent sampling methods.

(c) Explain why it is desirable to use the antithetic variates method in place of independent sampling for confidence interval estimation. Use your answer to part **(b)** to justify.