STOR 634
Exam: CWE   Year: 2011
You may appeal to any result proved in class without proof but state the result you use.

1. Let $\mathcal{F}$ be a σ-field on $\mathbb{R}$. Show that the Borel sigma-field $\mathcal{B}(\mathbb{R}) \subset \mathcal{F}$ if and only if every continuous function $f : \mathbb{R} \to \mathbb{R}$ is measurable with respect to $\mathcal{F}$. Thus $\mathcal{B}(\mathbb{R})$ is the smallest σ-field with respect to which all the continuous functions are measurable.

   Hint: Recall that a function f is continuous if and only if $f^{-1}(G)$ is open for every open set G.

2. For $i = 1, 2$, let $(\Omega_i, \mathcal{F}_i)$ be measure spaces with finite measures $\lambda_i << \nu_i$. Now consider the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2)$ with the two finite measures $\lambda_1 \times \lambda_2$ and $\nu_1 \times \nu_2$. Show that $\lambda_1 \times \lambda_2 << \nu_1 \times \nu_2$ and further
$$\frac{d\lambda_1 \times \lambda_2}{d\nu_1 \times \nu_2}(\omega_1, \omega_2) = \frac{d\lambda_1}{d\nu_1}(\omega_1)\frac{d\lambda_2}{d\nu_2}(\omega_2) \qquad a.e.\ \nu_1 \times \nu_2$$

3. Let F_0 be a distribution function on the real line. Define a sequence of functions F_n recursively for $n = 1, 2, \ldots$ by
$$F_n(x) = \int_{-\infty}^{x} [F_{n-1}(t) - F_{n-1}(t-1)]dt, \qquad x \in \mathbb{R}.$$
   Prove that F_n is a distribution function for all $n \geq 1$.

4. Consider the measure space $([0,1], \mathcal{B}([0,1]))$ equipped with the Lebesgue measure and consider the sequence of functions $\{f_n\}_{n\geq 3}$
$$f_n(x) = \frac{n}{\log n}\mathbb{1}_{A_n}(x)$$
   where the set $A_n = [0, 1/n]$. Show that this sequence is uniformly integrable.

5. Let X_i be iid ± 1 valued random variables with
$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$$
   Let $S_n = \sum_{i=1}^{n} X_i$.
   (a) For $a, b > 0$, let $T_{-a,b} = \inf\{n : S_n = -a \text{ or } b\}$. By setting up the appropriate martingale, find $\mathbb{P}(S_{T_{-a,b}} = -a)$. Justify all the steps you need to use the Optional Sampling theorem (most importantly, showing that $T_{-a,b} < \infty$ almost surely).
   (b) Now consider the case $b = a$ and for simplicity let $T_a = T_{-a,a}$. Fix $\lambda > 1$ and find an appropriate functions $\phi_n(\lambda)$ such that the sequence $\{M_n(\lambda)\}_{n\geq 1}$ defined as
$$M_n(\lambda) = \frac{\exp(\lambda S_n)}{\phi_n(\lambda)}$$
   is a martingale.
   (c) Assume that S_{T_a} is independent of T_a. Use the optional sampling theorem to calculate
$$\mathbb{E}(\frac{1}{\phi_{T_a}(\lambda)})$$