DEPARTMENT OF STATISTICS AND OPERATIONS RESEARCH
OPERATIONS RESEARCH DETERMINISTIC QUALIFYING EXAMINATION

August 16, 2011
9:00 am - 1 pm

General Instructions

This examination is closed-book, and consists of three questions. Answer all three as clearly and concisely as you are able. Use of the internet and/or mobile devices is not permitted.

Question 1 (25 points) The following linear program involves the making of four products — represented by x_1, x_2, x_3 and x_4 — maximizing profits in dollars, and subject to three resource constraints:

$$\begin{array}{rcccccccl} \max z = & 3x_1 & + & x_2 & + & 4x_3 & + & x_4 & \\ & 6x_1 & + & 3x_2 & + & 5x_3 & + & 4x_4 & \leq 25 \\ & 3x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & \leq 15 \\ & 3x_1 & + & 4x_2 & + & 5x_3 & + & 2x_4 & \leq 20 \\ & x_1 \geq 0 & & x_2 \geq 0 & & x_3 \geq 0 & & x_4 \geq 0 & \end{array}$$

This LP was solved, resulting in the following basic tableau:

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
1	0	2	0	1	1/5	0	3/5	?
0	1	−1/3	0	2/3	1/3	0	−1/3	?
0	0	0	0	−1	−2/5	1	−1/5	?
0	0	1	1	0	−1/5	0	2/5	?

a. Give the basis, basis matrix and inverse, and associated basic feasible solution and objective function values.

b. What is the smallest value that the objective coefficient of the third item (currently 4) can take and still have x_3 in the optimal basis? For this smallest such value, what will be the associated solution and objective function value?

c. What is the largest value of the first resource (currently 25) for which the current basis is optimal? For values slightly larger than this value, what will the new optimal basis be?

d. Suppose we can purchase a "package" of resources: 3 units of resource 1, 7 units of resource 2, and 4 units of resource 3, at a total cost of \$5. Should you purchase any of this package? Why or why not?

Question 2 (50 points) Recall that a subset C in Euclidean space is said to be *bounded* if there exists a positive real number M such that every $x \in C$ satisfies $\|x\| \leq M$, and *unbounded* otherwise.

For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, let

$$S = \{x \in \mathbb{R}^n \mid Ax \geq b,\ x \geq 0\}$$

and

$$T = \{y \in \mathbb{R}^m | A^T y \leq c,\ y \geq 0\}.$$

a. Construct a *single* LP whose solution tells whether or not the set S is a bounded set.

b. Suppose that both S and T are nonempty. Prove that at least one of the two sets S and T is unbounded.

c. Is it possible for S and T to be empty simultaneously? Prove or give a counterexample.

d. Is it possible that one of the two sets S and T is empty and the other is nonempty and bounded? Prove or give a counterexample.

e. Is it possible for S and T to be unbounded simultaneously? Prove or give a counterexample.

Question 3 (25 points) A firm uses m raw materials to manufacture n products. A pound of product j earns a profit of p_j dollars. Each raw material comes in two grades, I and II. A pound of product j requires a_{ij}^I pounds of grade-I raw material i and a_{ij}^{II} pounds of grade-II raw material i. There are available r_i^I (r_i^{II}) pounds of grade-I (grade-II) raw material i. Unused grade-I (grade-II) raw material i can be sold for s_i^I (s_i^{II}) dollars per pound.

a. Formulate an LP to determine the product mix that maximizes the firm's profit.

b. Suppose grade-I raw material i can be *downgraded*, that is, used in place of the equivalent grade-II raw material i, at no additional cost. Formulate another LP to determine the optimal product mix when downgrading is allowed.

c. Suppose that in addition to downgrading, it is also possible to *upgrade* a II-grade raw material to the equivalent grade-I raw material. However, it costs u_i dollars per pound to upgrade type i raw material. Modify the LP in (b) to determine the optimal product mix when both upgrading and downgrading are allowed.