STOR 635, CWE 2011-12

1. (*15 points*) Let $\xi_1, \xi_2, \cdots$ be independent with $I\!E(\xi_m) = 0$ and $var(\xi_m) = \sigma_m^2 < \infty$. Let $s_n^2 = \sum_{m=1}^n \sigma_m^2$ and $S_n = \sum_{m=1}^n \xi_m$. Show that $S_n^2 - s_n^2$ is a martingale.

2. (*20 points*) Suppose $\{X_n^1\}$ and $\{X_n^2\}$ are supermartingales with respect to some filtration $\{\mathcal{F}_n\}$ and N is a stopping time such that $X_N^1 \geq X_N^2$ a.s. Show that

$$Y_n = X_n^1 1_{N>n} + X_n^2 1_{N\leq n}$$

is a supermartingale.

3. (*15 points*)

a. (*5 pts*) Let μ be a probability measure on the infinite product space $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$. Say what it means for μ to be a product measure.

b. (*10 pts*) Let $\{X_k\}_{k\geq 1}$ be a sequence of $\mathbb{R}$ valued independent random variables, given on a probability space $(\Omega, \mathcal{F}, I\!P)$. Let $X = (X_1, X_2, \cdots)$. Let μ be the probability distribution of X. Show that μ is a product measure.

4. (*20 points*)
Let $0 \leq X_1 \leq X_2 \cdots$ be random variables such that $EX_n = an^\alpha$, with $a, \alpha > 0$, and $var(X_n) \leq Bn^\beta$ with $\beta < 2\alpha$. Show that $X_n/n^\alpha \to a$ a.s.

5. (*15 points*) Prove the "conditional" Holder's inequality: Let X, Y be random variables on $(\Omega, \mathcal{F}, I\!P)$ such that for some $p, q > 1$, $I\!E|X|^p < \infty$ and $I\!E|Y|^q < \infty$. Let $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$. Show that

$$(I\!E(|XY| \mid \mathcal{G})) \leq [I\!E(|X|^p \mid \mathcal{G})]^{1/p} [I\!E(|Y|^q \mid \mathcal{G})]^{1/q}, \quad a.s.$$

6. (*15 points*) Let $\mu_n \sim N(a_n, 1)$, where a_n is a real sequence. Show that the sequence $\{\mu_n\}$ is tight iff a_n is a bounded sequence.