STOR 642
Comprehensive Written Examination
August 2011

This test consists of two questions.
This is a closed book exam.
Explain your answers in detail.
The duration of the exam is 2 hours.
The relative weights are given in the parentheses.

Problem 1. Patients arrive at a clinic one by one. The nth patient at a clinic is scheduled to arrive time nd, $n = 0, 1, 2, \cdots$, where $d > 0$ is a fixed number. The nth patient fails to show up for his/her appointment with probability θ, independently of all other patients. There is a single doctor who sees the patients in a first come first served fashion. The service times are iid Exp(μ). Let $X(t)$ be the number of customers in the clinic at time t. Let X_n be the number of patients in the clinic just before the nth scheduled arrival time, and Y_n be the number of patients in the clinic as seen by the n patient who actually arrives.

1. (5) Show that the following limit exists:
$$p_j = \lim_{n\to\infty} P(X_n = j), \quad j \geq 0.$$
Compute it.

2. (5) Show that the following limit exists:
$$q_j = \lim_{n\to\infty} P(Y_n = j), \quad j \geq 0.$$
Compute it.

3. (2) Define a Markov regenerative process. Show that $\{X(t), t \geq 0\}$ is a Markov regenerative process.

4. (5) Show that the following limit exists:
$$r_j = \lim_{t\to\infty} P(X(t) = j), \quad j \geq 0.$$
Compute it.

5. (3) What is the relationship among the p_j's, q_j's and r_j's?

6. (5) Suppose the waiting costs of the patients is w per unit time, and the idle time of the doctor costs h per unit time. Compute the long run cost per unit time as a function of d.

Problem 2. (You may use the fact given at the end of this problem if needed.)

1. (1) Define a standard Brownian.
2. (2) Define a renewal process. Define transience and recurrence of a renewal process. Give the criteria for transience and recurrence.
3. (4) Let $\{B(t), t \geq 0\}$ be a standard Brownian motion. Define $S_0 = 0$ and

$$S_{n+1} = \min\{t > S_n : B(t) \in \{B(S_n) - 1, B(S_n) + 1\}\},\ n \geq 0.$$

   Let

$$N(t) = \sup\{n \geq 0 : S_n \leq t\}. \tag{1}$$

   Show that $\{N(t), t \geq 0\}$ is a renewal process. Is it transient or recurrent?

4. (4) State and prove the almost sure version of the elementary renewal theorem for a recurrent renewal process.
5. (4) Compute $\lim_{t\to\infty} N(t)/t$, where $N(t)$ is as defined in Equation 1.
6. (2) Compute $E(\int_0^{S_1} B^2(u)du | B(0) = 0)$.
7. (1) Define a renewal reward process.
8. (4) Define $R(t) = \int_0^t [B(u) - B(T_{N(u)})]^2 du$. Show that $\{R(t), t \geq 0\}$ is a renewal reward process.
9. (3) Compute $\lim_{t\to\infty} R(t)/t$.

Fact: Let $\{B(t), t \geq 0\}$ be a standard Brownian motion, $T_{ab} = \min\{t \geq 0 : B(t) \in \{a, b\}\}$. Let $c(x) = E(\int_0^{T_{ab}} f(B(u))du | B(0) = x)$ for $a \leq x \leq b$. Then c satisfies the differential equation

$$\frac{1}{2}c''(x) = -f(x),$$

with boundary condition

$$c(a) = c(b) = 0.$$