## Statistics 654 Comprehensive Written Exam August, 2012

- 1. Let U, V be random variables with finite second moment, and let  $SD(\cdot)$  denote the usual standard deviation.
- a. Show that  $SD(U + V) \leq SD(U) + SD(V)$ .
- b. Show that  $\mathbb{E}|U| + \mathbb{E}|V| \leq \sqrt{2}(\mathbb{E}U^2 + \mathbb{E}V^2)^{1/2}$
- 2. Let  $\mathcal{P} = \{f_{\theta}(x) : \theta \in \Theta\}$  be a family of probability mass functions for a discrete random variable  $X \in \mathcal{X}$ .
- a. Define what it means for a statistic  $T: \mathcal{X} \to \mathbb{R}^d$  to be sufficient for  $\theta$ .
- b. Carefully state the Factorization Theorem for sufficient statistics in the discrete setting above.
- c. The Factorization Theorem is of an "if and only if" form. Prove one direction of the theorem.
- 3. Let X denote an i.i.d. sample  $X_1, \ldots, X_n$  from the  $\mathcal{N}(\mu, \sigma^2)$  distribution.
- a. What are the maximum likelihood estimates of  $\mu$  and  $\sigma$  based on X? You may write down the estimates without deriving them. Are these estimates unbiased?
- b. Establish the numerical identity  $\sum_{i=1}^{n} (x_i \theta)^2 = \sum_{i=1}^{n} (x_i \overline{x})^2 + n(\overline{x} \theta)^2$ , valid for real numbers  $x_1, \ldots, x_n, \theta$ .

Suppose now that we wish to test the hypothesis  $H_0: \mu \leq \mu_0$  vs.  $H_1: \mu > \mu_0$  for a fixed number  $\mu_{0_1}$  and that the population variance  $\sigma^2$  of our sample is unknown.

- c. Carefully identify the natural t-statistic T(x) for this test. Does one reject  $H_0$  for large or small values of T?
- d. Show as carefully as you can that testing based on T(x) is equivalent to a likelihood ratio test
- e. Define the p-value associated with an observed statistic T(x), and show that it has a simple, closed form.

Suppose now that we wish to test the hypothesis  $H_0: \mu = \mu_0$  vs.  $H_1: \mu > \mu_0$ , again assuming that the variance  $\sigma^2$  is unknown.

- f. Let  $\alpha \in (0,1)$ . Identify a one-sided level- $\alpha$  test for  $H_0$  vs.  $H_1$ . Be sure to establish that the level of the test is indeed  $\alpha$ .
- g. Use the test in the part f to find a  $1-\alpha$  confidence interval for  $\mu$ . Carefully identify the steps in your argument.