STOR 762
Comprehensive Written Exam
August 15, 2012

- This exam consists of 3 questions on 2 pages.
- This is a closed book exam.
- You are not allowed to use a calculator or a cell phone throughout the exam.
- Explain your answers in detail.
- You may need the following information:
  - If $X \sim$Uniform(a, b), then $\mathrm{E}[X] = (a + b)/2$ and $\mathrm{Var}(X) = (b - a)^2/12$.
  - Suppose $\{X_i\}_{i=1}^n$ is a sequence of random variables and $\{Y_i\}_{i=1}^n$ is another sequence of random variables. Then,

$$\mathrm{Cov}\left(\sum_{i=1}^n \alpha_i X_i, \sum_{j=1}^m \beta_j Y_j\right) = \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j \mathrm{Cov}\left(X_i, Y_j\right),$$

    where α_i's and β_j's are constants.

Problem 1. (33 points) Consider a queueing system with six parallel identical servers. Customers arrive to this queueing system according to a Poisson process and join a single queue. The service time for each customer is independent and identically distributed. Suppose that you have conducted a single simulation run of this queueing system and collected the following data: $\{Q(t), 0 \leq t \leq T\}$, where $Q(t)$ is the number of jobs in the system at time t and T is the total simulated time. Suppose also that the simulation started in a state where all servers are busy but there are no other jobs waiting in the queue. You are told that the process reaches steady state at around $t = \tau$, where τ is much smaller than T.

(a) (11 points) Explain how you would apply the batch means method with b batches to construct a 95% confidence interval on the steady-state mean number of busy servers. Be specific about the upper and lower bounds of the confidence interval, i.e., express them as functions of $\{Q(t), 0 \leq t \leq T\}$, T, τ, etc.

(b) (11 points) Suppose that the confidence interval that you obtained in part **(a)** is too large for the precision that you need. Explain what you would do to reduce the confidence interval length to approximately one-third of its size without compromising from the coverage of the interval.

(c) (11 points) Suppose that during the given simulation run, i.e., during $[0, T]$, the number in the system hits zero twenty times and $Q(T) = 7$. Explain how you would use the regenerative method to estimate the long-run average fraction of time the queue is empty. Be specific about your estimator, i.e., express it in terms of $\{Q(t), 0 \leq t \leq T\}$, T, etc.

Problem 2. (35 points) Consider two first-order moving average processes $\{X_i\}_{i\geq 1}$ and $\{Y_i\}_{i\geq 1}$, which are defined by

$$\begin{aligned} X_i &= \alpha\epsilon_{i-1} + \epsilon_i, \text{ for } i = 1, 2, \ldots, \\ Y_i &= \beta\phi_{i-1} + \phi_i, \text{ for } i = 1, 2, \ldots, \end{aligned}$$

where $|\alpha| < 1$ and $|\beta| < 1$ are given constants, ϵ_i's are independent and uniformly distributed over the interval $(0, a)$ and ϕ_i's are independent and uniformly distributed over the interval $(0, b)$. Our objective is to use a simulation experiment to estimate the long-run average difference between the means of these two processes.

For this purpose, we perform 10 replications each of length n observations for both processes and use common random numbers for variance reduction. Assume that the inverse-transform method is used for the generation of all uniform variates. Suppose that $X_i^{(j)}$ and $Y_i^{(j)}$ represent the ith observation from replication j for process $\{X_i\}_{i\geq 1}$ and $\{Y_i\}_{i\geq 1}$, respectively. Let also $\{\epsilon_i^{(j)}\}_{i=0}^n$ be the sequence of uniform variates used in replication j for process $\{X_i\}_{i\geq 1}$ and define $\{\phi_i^{(j)}\}_{i=0}^n$ be the sequence of uniform variates used in replication j for process $\{Y_i\}_{i\geq 1}$.

(a) (5 points) Obtain the covariance of $\epsilon_i^{(1)}$ and $\phi_i^{(1)}$ for $i = 1, 2, \ldots, n$.

(b) (10 points) Obtain the covariance of $X_i^{(1)}$ and $Y_k^{(1)}$ for all $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n$.

(c) (10 points) Let $\bar{X}^{(j)}$ and $\bar{Y}^{(j)}$ denote the sample mean from replication j for processes $\{X_i\}$ and $\{Y_i\}$, respectively. Obtain the covariance between $\bar{X}^{(1)}$ and $\bar{Y}^{(1)}$.

(d) (10 points) Using your answer to part **(c)**, obtain the correlation coefficient between $\bar{X}^{(1)}$ and $\bar{Y}^{(1)}$. Based on this correlation coefficient, for what values of α, β, a, and b, the common random numbers method is expected to be most efficient?

Problem 3. (32 points) Consider a random variable X that has a geometric distribution with parameter $p \in (0, 1)$, i.e., $\Pr\{X = i\} = p(1-p)^i$ for $i = 0, 1, \ldots$.

(a) (11 points) Prove that the following algorithm can be used to generate geometric random variates exactly:

Step 1. Generate a random number U.

Step 2. Return $X = \lfloor \ln(U)/\ln(1-p) \rfloor$, where $\lfloor x \rfloor$ is the floor function that returns the largest integer smaller than or equal to x.

(b) (11 points) Propose another exact algorithm for generating a geometric random variate.

(c) (10 points) Compare the algorithm that you proposed in part **(b)** with the algorithm provided in part **(a)** in terms of their computational efficiency.