DEPARTMENT OF STATISTICS AND OPERATIONS RESEARCH
OPERATIONS RESEARCH DETERMINISTIC QUALIFYING EXAMINATION

August 13, 2012
9:00 am - 1 pm

General Instructions

This examination is closed-book, and consists of four questions. Answer all four as clearly and concisely as you are able. Use of the internet and/or mobile devices is not permitted.

Question 1 (25 points; STOR 612) Howie Cramms need to obtain at least G grade points on the final exams for n courses. He needs to take all n exams, but it is fine if he gets very few, or even 0 points in some of them, as long as the *total* number of points he gets is at least G. If Howie studies h hours on Course $\#j$, $h = 1, \ldots, H$, he can obtain $p(h, j)$ grade points for that course.

What is the smallest total number of hours Howie needs to study obtain at least G grade points?

a. Give a general dynamic programming method for solving this. The total number of operations that your algorithm takes should be polynomial in n and H.

b. Solve this problem where Howie needs to get 19 points on four courses, and the number of points obtained per hour studied is given by the following table ($H = 4$):

hours studied	Course #1	#2	#3	#4
0	0	0	0	0
1	3	5	2	6
2	5	5	4	7
3	6	6	7	9
4	7	9	8	9

c. What is the maximum number of points that can be made by spending the number of hours found in part b?

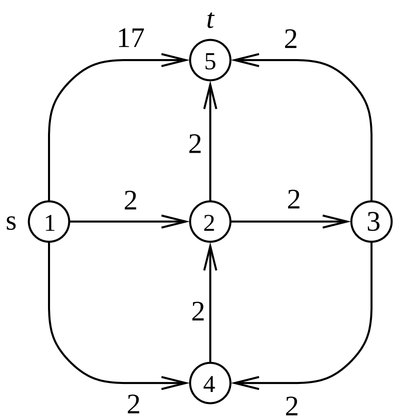
Question 2 (25 points; STOR 612) In shortest path problems there is often a *turn penalty* for making certain turns in the path (for example, left turns in heavy traffic) and the shortest path should take these turn penalties into account. In particular, a shortest path problem with turn penalties will have as input directed network $G = (N, A)$, arc distances $(c_{ij} : (i, j) \in A)$, and turn penalties of the form

$$t_{ef} \text{ for each pair } e, f \text{ with } e = (i, j) \text{ and } f = (j, k).$$

The length of the path is now the sum of the arc lengths of the path and the turn penalties between successive pairs of arcs in the path.

a. Let G be a rectilinear graph, that is, all arcs are north-south or east-west and so all turns are right, left, or straight through (no U-turns). Show how a shortest path problem with turn penalties can be transformed into a standard shortest path problem. (Hint: You can enter or leave an intersection in one of four ways. Put in a node for each of these.)

b. Apply this to the following shortest path problem with turn penalties 1 for each right turn, 2 for each straight-through, and 15 for each left turn. (Assume that there are no turn costs at s or t or for the four curves.) Give the associated transformed network and shortest path. (You don't need to show the method of solution here). Then show the associated "path" in the original graph.



Question 3 (20 points; STOR 614) Let A be an $m \times n$ matrix and $p \in \mathbb{R}^n$. Consider the following linear program in which $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$ are variables:

$$\begin{array}{ll} \min & t \\ \text{s.t.} & Ay = 0, \\ & p^T y - t = -1, \\ & y \geq 0, \ t \geq 0. \end{array} \qquad (P)$$

a. Write down the dual LP (D). For full credit, use $x \in \mathbb{R}^m$ as the dual variable associated with the constraint $Ay = 0$, and use $s \in \mathbb{R}$ as the dual variable associated with the constraint $p^T y - t = -1$.

b. Show that both (P) and (D) have optimal solutions.

c. Let v^* be the common optimal value of (P) and (D). Show that it satisfies $0 \leq v^* \leq 1$.

d. Show that in fact v^* is equal to either 0 or 1.

Question 4 (30 points; STOR 614) Consider the primal-dual pair of LPs

$$(P)\quad \begin{array}{ll} \max & c^T x \\ s.t. & Ax \le b \end{array} \qquad\qquad \begin{array}{ll} \min & b^T y \\ s.t. & y \ge 0 \\ & A^T y = c. \end{array} \quad (D)$$

Here $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and A is an $m \times n$ matrix with linearly independent columns. Also consider the following pair of problems

$$(P')\quad \begin{array}{ll} \min & y_0^T z \\ s.t. & z \in (z_0 + L) \cap \mathbb{R}^m_+ \end{array} \qquad\qquad \begin{array}{ll} \min & z_0^T y \\ s.t. & y \in (y_0 + L^\perp) \cap \mathbb{R}^m_+ \end{array} \quad (D')$$

Here $y_0,\ z_0 \in \mathbb{R}^m,\ L$ is a subspace in $\mathbb{R}^m,$ and $L^\perp$ is its *orthogonal complement* defined as

$$L^\perp = \{y \in \mathbb{R}^m \mid y^T z = 0 \text{ for all } z \in L\}.$$

In answering the following questions, suppose A, b, c are given, and assume that (P) and (D) both have nonempty feasible sets.

a. Find $y_0,\ z_0,\ L$ such that (D) and (D') represent exactly the same LP. (Hint: let L be the *range space* of A. Recall that the range space of a matrix B is the set of linear combinations of columns of B, and that the orthogonal complement of the range space of B is the nullspace of B^T.)

b. In this and the remaining questions, let $y_0,\ z_0,\ L$ be as chosen in part a. Show that the feasible solutions of (P) and (P') are in one to one correspondence. Describe such correspondence precisely.

c. Show how the objective value of any given feasible solution of (P) is related to the objective value of the corresponding feasible solution of (P'). Do the same for (D) and (D').

d. Let v_z be the optimal value of (P') and v_y be the optimal value of (D'). How are these two related?