STOR 635

Exam: CWE Year: 2013

You may appeal to any result proved in class without proof unless you are specifically asked to provide a complete proof. State any result you use. All questions are worth the same number of points. Attempt all questions.

1. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For any set $B \in \mathcal{B}(\mathbb{R})$ and real number $x \in \mathbb{R}$, define the set $B - x := \{y - x : y \in B\}$. Let μ_1, μ_2 be two probability measures on $\mathbb{R}$. Define the set function $\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{R}_+$ as
$$\mu(B) := \int_{\mathbb{R}} \mu_1(B - x) d\mu_2(x), \qquad B \in \mathcal{B}(\mathbb{R}).$$
Show that μ defined above is a probability measure and further the above definition is commutative in the sense that for any set $B \in \mathcal{B}(\mathbb{R})$,
$$\int_{\mathbb{R}} \mu_1(B - x) d\mu_2(x) = \int_{\mathbb{R}} \mu_2(B - x) d\mu_1(x).$$

2. Fix a probability mass function $\mathbf{p} := \{p_k\}_{k \geqslant 0}$ and let $\{Z_n\}_{n \geqslant 0}$ with $Z_0 = 1$ be a branching process with offspring distribution $\mathbf{p}$. More precisely, let $\{\xi_{i,j} : i, j \geqslant 1\}$ be **i.i.d** with distribution $\mathbf{p}$. Now define the sequence $\{Z_n\}_{n \geqslant 0}$ recursively with $Z_0 = 1$ and let
$$Z_n := \sum_{j=1}^{Z_{n-1}} \xi_{n,j} \qquad n \geqslant 1,$$
with the understanding that if $Z_{n-1} = 0$ then $Z_n = 0$. Define the probability generating function of $\mathbf{p}$ as
$$\phi(s) := \sum_{k=0}^{\infty} s^k p_k, \qquad s \in [0, 1]$$
Suppose there exists a unique $0 < \rho < 1$ such that $\phi(\rho) = \rho$. Show that
$$\rho = \mathbb{P}(Z_n = 0 \text{ eventually })$$
Hint: You may use the following fact: Define the events
$$A := \{\omega | Z_n(\omega) \to \infty \text{ as } n \to \infty\}, \qquad B := \{\omega | Z_n(\omega) = 0 \text{ eventually }\}.$$
You may assume that $\mathbb{P}(A \cup B) = 1$. Now study the sequence $\{\rho^{Z_n}\}_{n \geqslant 0}$.

3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{F}$ be two subcollections of events. Recall that we say that $\mathcal{C}_1, \mathcal{C}_2$ are independent if for every pair of sets $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$,
$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$
Now suppose $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{F}$ are fields which are independent. Show that the sigma fields generated by these, $\sigma(\mathcal{C}_1)$ and $\sigma(\mathcal{C}_2)$ are also independent. **Give a complete proof.**

4. Consider the space representing an infinite sequence of coin flips, namely $\Omega := \{H, T\}^\infty$, with the associated sigma field $\mathcal{F}$ generated by finite dimensional rectangles. For $0 \leqslant p \leqslant 1$ let $\mathbb{P}_p$ denote the probability measure on $(\Omega, \mathcal{F})$ corresponding to flipping a coin an infinite number of times with probability of H being p and probability of T being $q = 1 - p$ at each flip. Show that these measures are mutually singular in the sense that for any $p_1 \neq p_2$, there exist sets A_{p_1} and A_{p_2} which are disjoint namely $A_{p_1} \cap A_{p_2} = \emptyset$ but $\mathbb{P}_{p_i}(A_{p_i}) = 1$ for $i = 1, 2$.

5. (a) Let $(X_i)_{i \geqslant 1}$ be a sequence of finite random variables and let $\mathcal{T}$ be their tail sigma-field. Let $S_n = \sum_{i=1}^n X_i$. Define the event,

$$A = \left\{\omega : \lim_{n \to \infty} S_n(\omega) \text{ exists and is finite}\right\}.$$

Show that $A \in \mathcal{T}$. Now suppose $(X_i)_{i \geqslant 1}$ are **i.i.d.** with $\mathbb{E}(X_i) = 0$ and $0 < \mathbb{E}(|X_i|^3) < \infty$. Show that $\mathbb{P}(A) = 0$.

(b) Let $(X_i)_{i \geqslant 1}$ be an iid sequence of random variables having exponential distribution with mean $\mu = 1$ namely, the probability density function is given by

$$f(x) := e^{-x}, \qquad x \geqslant 0$$

Let $S_n = \sum_{i=1}^n X_i$. Fix any $a > 1$. Show that for any $n \geqslant 1$ we have

$$\mathbb{P}(S_n > na) \leqslant \exp\left(-n\left[(a-1) - \log a\right]\right)$$