STOR641 - Comprehensive Written Exam - August 2013

This test consists of two questions, each with multiple parts. The relative weights are specified following each part. The duration of the exam is two hours and you are NOT allowed to refer to any notes, books etc.

Good luck!

1− Passengers arrive at a bus depot according to a Poisson process with rate $\lambda > 0$. An officer visits the bus depot at time intervals that follow and independent Poisson process with rate $\mu > 0$. If the officer finds that there are at least $k > 0$ passengers waiting at the bus depot, he immediately calls enough buses to transport all the waiting customers. (Buses are assumed to arrive and leave with the passengers instantaneously.)

(a) Model this system as a continuous-time Markov chain.**(10 points)**

(b) Do we need any conditions for this Markov chain to be positive recurrent? If no, prove that the chain is positive recurrent. If yes, find the necessary and sufficient conditions under which the Markov chain is positive recurrent. Then, assuming positive recurrence, find the limiting distribution. **(15 points)**

(c) Consider a passenger who arrives to an empty bus depot. What is the expected time until this passenger leaves with a bus? **(7 points)**

(d) Let N denote the number of passengers who leave with buses at a random departure instant. What is the probability distribution of N? **(8 points)**

2− Visitors arriving to the first floor of a three-story building use the elevator to get to the second or third floors. Visitors who are already on the second and third floors use stairs so that no visitor gets on the elevator on those floors. It takes one period of time for the elevator to travel up or down one floor. Visitors are assumed to get on and off the elevator and press the button that indicates their destination instantaneously. Once the visitors leave the elevator on the second or third floor, the elevator goes back to the first floor even if there is no one waiting and sits there idle until someone arrives.

At the beginning of any period, if the elevator is at the first floor, it lets in visitors who have already queued up to get on the elevator. The elevator has a capacity of K. So, if there are more than K visitors in the queue, the first K in the queue are admitted and the rest continue to wait for the next trip. When the visitors get on the elevator, each visitor indicates the floor s/he would like to visit. If none of the visitors would like to visit the third floor, then the elevator makes a round trip to the second floor. If there is at least one visitor who would like to visit the third floor, then the elevator makes a round trip to the third floor. (The elevator may stop at the second floor on the way but because visitors are assumed to get off instantaneously, the elevator continues its trip without any delay.) If, at the beginning of any period, the elevator is at the second or third floor, then any visitors on the elevator who would like to get off leave and the elevator continues its trip.

At the end of each period, new visitors arrive to get on the elevator at the first floor and they line up in a first-come-first-served fashion. The probability that there are j arrivals is β_j for $j \in \{0, 1, 2, \dots\}$ and this probability is independent of the system state and the arrivals at other periods. Each visitor would like to visit the second floor with probability α and the third floor with probability $1 - \alpha$ independently of other visitors and the system state. We assume that $0 < \alpha < 1$.

(a) We would like to answer a number of questions for this system. These questions are given below. Take a quick look at these questions first without attempting to answer them. Then, model this system as a discrete-time Markov chain, which can be used to answer these questions. In particular, give a clear description of the state space and transition probabilities. **(15 points)**

(b) Write the balance equations. **(15 points)**

For the rest of the questions, assume that the Markov chain is positive recurrent. Give your answers in terms of π_s for $s \in S$ where π_s is the limiting probability that the Markov chain is in state s. Do not solve the balance equations.

(c) Give an expression for the steady-state expected elevator queue length at the beginning of a period (right before any visitor gets on the elevator). **(10 points)**

(d) Even though visitors arrive as a batch at the end of each period, they need to queue up one by one to get on the elevator. Consider the visitor who is the first in a batch, which arrived in a random period. What is the steady-state expected time until this visitor gets on the elevator? **(10 points)**

(e) What is the steady-state probability that a visitor who finds the elevator queue empty finds the elevator is about to arrive to the second floor coming from the third floor? **(10 points)**