STOR 642
Comprehensive Written Examination
9:00am-1:00pm, August 15, 2013

This test consists of three questions.
This is a closed book exam. No communication devices of any kind are allowed.
Explain your answers in detail.
The duration of the exam is 2 hours.
The problem weights are given in parentheses.

Problem 1. Consider a single server queueing system where customers arrive according to a PP(λ). Customers who find the system empty upon arrival are called type 0 customers and require exp(μ_0) service times, while the others are called type 1 customers and require exp(μ_1) service times. Assume that $\mu_1 < \mu_0$ and all service are iid. Let $X(t)$ be the number of customers in the system at time t and $Y(t)$ be 0 if the server is idle or serving a type 0 customer at time t, and 1 otherwise.

1. (6) Show that $\{(X(t), Y(t)), t \geq 0\}$ is a CTMC by constructing its rate diagram.

2. (6) What is the condition of stability?

3. (6) Assume stability and let

$$p(i,j) = \lim_{t\to\infty} P(X(t) = i, Y(t) = j), \quad i \geq 0,\ j = 0, 1.$$

Write the balance equations satisfied by these limiting probabilities. Do not solve them.

4. (6) Show that

$$p(0,0) = (1 - \rho_1)/(1 + \rho_0 - \rho_1),$$

where $\rho_i = \lambda/\mu_i$, $i = 0, 1$.

5. (6) What is the limiting probability that the server is busy serving a customer of type 0? (Hint: You may use part 4 above.)

Problem 2. Suppose the lifetime of a machine is an exp(μ) random variable. A repair-person visits a machine at random times as follows. If the machine is working when the repair-person visits, he simply goes away and returns after d amount of time, where d is a fixed positive constant. If the machine is down when the repair-person visits, he repairs the machine in r amount of time, where r is a fixed positive constant. Then he goes away and returns after d amount of time. The machine is as good as new after the repair is complete. Let $N(t)$ be the number of repair completions during $(0, t]$. Furthermore, let $X(t)$ be 0 if the machine is working at time t, 1 if it is down at time t, and 2 if it is under repair at time t. Suppose a repair has just completed at time 0.

1. (6) Is $\{N(t), t \geq 0\}$ a renewal process? Justify your answer.

2. (6) Compute

$$\lim_{t \to \infty} \frac{N(t)}{t}.$$

3. (6) Is $\{X(t), t \geq 0\}$ a semi-Markov process? Why or why not? Describe its kernel if it is an SMP.

4. (6) Is $\{X(t), t \geq 0\}$ a regenerative process? Why or why not?

5. (6) Is $\{X(t), t \geq 0\}$ a Markov-regenerative process? Why or why not?

6. (8) Compute the limiting distribution of $\{X(t), t \geq 0\}$.

Problem 3. Let $X(t)$ be the temperature (in degrees Fahrenheit) of a furnace at time t. We can control the temperature by turning it on and off. When the furnace is turned on the temperature behaves like a Brownian motion with drift parameter 1 degree per minute, and variance parameter 1. When it is down, it behaves like a Brownian motion with drift parameter -2 degrees per minute, and variance parameter 2. The aim is to keep the temperature as close to c degrees as possible. This is achieved by turning the furnace off when the temperature rises to $c + u$ degrees (where u is a fixed positive number.) Once the furnace is off it is kept off until the temperature drops to $c - d$ degrees (where d is a fixed positive number). At that time the furnace is turned on and kept on until the temperature rises to $c + u$ degrees. This control policy continues forever.

1. (8) Let α be the chatter rate of the furnace, defined as the number of times the furnace switches from off to on per hour in the long run. Compute α as a function of u and d.

2. (8) What is the expected temperature of the furnace in the long run?

3. (8) What fraction of the time is the furnace on in the long run?

4. (8) What values of u and d would you choose if the aim is to keep the long run average temperature at c degrees, keep the chatter rate bounded above by 10 per hour, and keep the temperature range $(u + d)$ to a minimum?

Hint: You may use the following fact: Let $\{B(t), t \geq 0\}$ be a BM(μ, σ), with $B(0) = 0$ and $\mu > 0$. Let $a > 0$ and define $T = \min\{t \geq 0 : B(t) = a\}$. Then

$$E(T) = \frac{a}{\mu}, \quad E\left(\int_0^T B(u)du\right) = \frac{a^2}{2\mu}.$$