Statistics 654 Comprehensive Written Exam
August, 2013

Instructions: Answer the following questions to the best of your ability. *Please show your work, and briefly explain your reasoning as necessary. Correct answers with no work/justification will not receive full credit.*

If you cannot answer a questions completely, write down, succinctly, the ideas you have for approaching the problem. Please write clearly.

1. Let U, V be random variables with finite second moment, and let $\mathrm{SD}(\cdot)$ denote the usual standard deviation. Find an inequality relating $|\mathrm{SD}(U) - \mathrm{SD}(V)|$ and $\mathrm{SD}(U - V)$.

2. Let X be a random variable with a CDF $F(\cdot)$ that is strictly increasing, and therefore invertible. What is the distribution of $F(X)$?

3. Define what it means for a family $\mathcal{P}$ of densities to be a scale family.

4. Define the notion of a pivot in the theory of confidence sets.

5. Let $U_1, \ldots, U_n$ be an i.i.d. sample from a density f in a scale family $\mathcal{P}$. Describe three essentially different pivots for this situation.

6. Let U, V be independent $\mathcal{N}(0,1)$ random variables. Are $U + V$ and $U - V$ independent? Explain your answer.

7. Let $X_1, \ldots, X_n$ be i.i.d. $\mathcal{N}(\theta, \sigma^2)$, where $\sigma^2 > 0$ is known.

a. Find the likelihood ratio tests of $\mathrm{H}_0 : \theta = \theta_0$ versus $\mathrm{H}_1 : \theta \neq \theta_0$, and express the test in a simple form. You may find it convenient to use the identity $\sum_{i=1}^n (u_i - v)^2 = \sum_{i=1}^n (u_i - \overline{u})^2 + n(\overline{u} - v)^2$.

b. Invert the test in part (a) to find a $1 - \alpha$ confidence interval for θ.

c. Is the test in part (a) unbiased? Justify your answer.

8. Let $X_1, \ldots, X_n$ be i.i.d. random variables taking values in $[0,1]$ and such that $EX_i = 1/2$. We are interested in upper bounds on the probability

$$\mathbb{P}\left(\frac{X_1}{X_1 + \cdots + X_n} \geq \frac{t}{n}\right) \tag{1}$$

for values of $t > 1$.

a. Find the expected value of $X_1/(X_1 + \cdots + X_n)$. (No extensive calculations are necessary.)

b. Find an upper bound on the probability in (1) using the Bounded Difference (McDiarmid) inequality. Show your work.

c. Find a better bound on the probability in (1).

9. Let $Y \sim \mathcal{N}(0, \Sigma)$ be a multi-normal random vector with covariance matrix Σ. Suppose that $EY_i^2 = 1$ for $1 \leq i \leq n$ and that every entry of $\Sigma - I$ is non-negative, where I denotes the $n \times n$ identity matrix. Let $\Phi(\cdot)$ be the CDF of the standard normal. Show that

$$\mathbb{E}\big(\Phi(Y_1) \cdots \Phi(Y_n)\big) \geq 2^{-n}.$$