DEPARTMENT OF STATISTICS AND OPERATIONS RESEARCH
OPERATIONS RESEARCH DETERMINISTIC QUALIFYING EXAMINATION

August 12, 2013
9:00 AM - 1 PM

General Instructions

This examination is closed-book, and consists of three questions. Answer all three as clearly and concisely as you are able. Use of the internet and/or mobile devices is not permitted.

Question 1. (33 points) We are given one product, and the following scenario. We have

- m supply points, the ith point having supply s_i.
- n demand points, the jth demand point having demand d_j.
- p transshipment points, which we abbreviate as tsp's.

Demands must be satisfied exactly. The amount of supply shipped out of supply point i cannot exceed s_i, but it may be less. To ship the product from supply point to demand point, it must go through exactly one tsp. One supplier may ship to several tsp's, and one demand point may receive product from several tsp's. The cost of shipping one unit of product from supply point i to tsp k is c_{ik}, of shipping one unit of product through tsp k is g_k, and of shipping one unit of product from tsp k to demand point j is h_{kj}.

Formulate each of the following optimization problems either as an LP or an MIP. Use the minimal number of variables and constraints to within a constant factor: for example, if a problem can be formulated with, say, $nm + nk$ constraints, then using twice as many is fine, in fact, preferable, if this makes the formulation cleaner. But using mnk will not receive full credit. Carefully describe the meaning of all variables and constraints. Try to model each problem as an LP, using integer variables only if you cannot model the problem otherwise.

For simplicity, define the *load* of tsp k to be the amount of product that is shipped through it.

a. (8 points) Formulate the problem of satisfying the demand of all demand points from available supply at the supply points at minimum cost.

b. (5 points) Modify the formulation of Part a to accommodate the restriction that the difference between the largest and smallest load should not be more than an upper bound U.

c. (5 points) Assuming the scenario of Part a, in addition model the following constraints: if the load of tsp k is positive, then it must be between lower and upper bounds L_k and U_k, where $0 < L_k \leq U_k$.

d. (5 points) Assuming the scenario of Part a, in addition model the following constraints: if the load of tsp k is positive, then this will incur a positive fixed cost f_k.

e. (5 points) Assuming the scenario of Parts a and d (but not of Parts b and c), in addition model the following constraints: tsp k can receive product from at most U_k suppliers, and can ship to at most V_k demand points.

f. (5 points) Assuming the scenario of Parts a, d and e (but not of the other parts), in addition model the following restriction: if the load of tsp 1 is at least a given lower bound L, then all of the remaining load from tsp 1 should be sent to demand point 1.

Question 2. (27 points) We are given functions $w_1, \ldots, w_n, b_1, \ldots, b_n$, which all map from nonnegative integers to nonnegative integers. All of them are increasing, and the w_i have the property that

$$w_i(x+1) - w_i(x) \geq \delta \text{ for all } x$$

where $\delta > 0$ is an integer.

a. (10 points) Solve the following optimization problem by dynamic programming:

$$\begin{aligned} \max \quad & \textstyle\sum_{i=1}^n b_i(x_i) \\ s.t. \quad & \textstyle\sum_{i=1}^n w_i(x_i) \leq \alpha \\ & x_i \in \{0, 1\} \end{aligned}$$

where α is a positive integer. Your algorithm should take $O(\alpha n)$ arithmetic operations, comparisons, and function evaluations.

b. (10 points) Solve the following optimization problem by dynamic programming:

$$\begin{aligned} \max \quad & \textstyle\sum_{i=1}^n b_i(x_i) \\ s.t. \quad & \textstyle\sum_{i=1}^n w_i(x_i) \leq \alpha \\ & x_i \geq 0, \text{ integer.} \end{aligned}$$

Your algorithm should take $O(\alpha^2 n/\delta)$ arithmetic operations, comparisons, and function evaluations.

c. (7 points) For the second part (where the x_i are nonnegative integers), define a directed graph, in which finding the longest path corresponds to computing the optimal solution. Give a formal proof of the equivalence.

Question 3. (40 points) Consider the nonempty polyhedron

$$P = \{x \in \mathbb{R}^n : \ Ax \leq b, \ x \geq 0\}.$$

where A is an $m \times n$ matrix and b is a column m-vector. For row n-vector α and scalar $\beta \in \mathbb{R}$, we say that an inequality $I(\alpha, \beta)$ of the form $\alpha x \leq \beta$ is a *valid inequality* for P if it is satisfied for all $x \in P$.

a. (15 points) **Prove:** $I(\alpha, \beta)$ is a valid inequality for P if and only if there exists a $y \in \mathbb{R}^m$ satisfying $yA \geq \alpha$, $yb \leq \beta$, $y \geq 0$. (**Hint**: Consider the LP to maximize $\{\alpha x : \ Ax \leq b, \ x \geq 0\}$.)

b. (25 points) Suppose $q \in \mathbb{R}^n \setminus P$. An inequality $I(\alpha, \beta)$ *separates* q *from* P if $I(\alpha, \beta)$ is valid for P, but $\alpha q > \beta$. We wish to find a "deepest" such separating inequality — one which maximizes the "infeasibility" $\alpha q - \beta$, subject to the normalizing constraints $|\alpha_i| \leq 1$ for $i = 1, \ldots, n$.

   Prove that the maximum infeasibility equals

$$\min\{\sum_j |q_j - x_j| : \ x \in P\}.$$

   (**Hint**: Construct an LP whose solution finds a deepest separating inequality and consider its dual.)