In this exam, we shall analyze the Gamma distribution $G(\alpha,\beta)$, which has a probability density function

$$p(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$$

for $x > 0$, where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is the Gamma function, and α and β are parameters. Suppose we observe $X_1, \ldots, X_n$ from the Gamma distribution with known α, and β is our parameter of interest.

1. Show that the Gamma distribution with known α is an exponential family.
2. An estimate for β (not necessarily unbiased!) is $\frac{1}{3}(X_1 + 2X_2)$. Use Rao-Blackwell Theorem to improve it.
3. Find the maximum-likelihood estimator T_n for β; as well as its expected value and variance. Is T_n a UMVUE?
4. Check the conditions of Cramér's Theorem on the MLE, and find the asymptotic distribution of $\sqrt{n}(T_n - \beta)$.
5. Find a function $g(\cdot)$ such that $\sqrt{n}(g(T_n) - g(\beta)) \xrightarrow{d} \mathcal{N}(0,1)$. Construct a 95% asymptotic confidence interval for β based on this limit.
6. Let $P = G(\alpha, \beta_0)$ and $Q = G(\alpha, \beta_1)$. Find the Hellinger affinity between P and Q : $\rho(P,Q)$.
7. In what follows, we shall focus on the maximum of Gamma variables.

   (a) Show the following lemma on the tail of Gamma integrals: For $0 < z \le 1$,

   $$\int_x^\infty t^{z-1}e^{-t}dt \le x^{z-1}e^{-x}.$$

   (b) Show that for $0 < \alpha \le 1$ and $\beta = 1$,

   $$\max_{1\le i\le n} X_i = O_p(\log n).$$

8. Suppose there are 10 Gamma distributions $G(\alpha, \beta_i), i = 1, \ldots, 10$, and tests are performed on each distribution: $H_{0,i} : \beta_i = b_i$. The p-values of the tests (sorted from low to high) are

   0.003, 0.007, 0.016, 0.022, 0.024, 0.033, 0.036, 0.092, 0.163, 0.345.

   Suppose the null p-values are independent. If we use the Benjamini-Hochberg procedure to control the false discovery rate at 0.05, which hypotheses will be rejected?