STOR 635, CWE 2013-14

1. (*15 points*) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$. Let $\mu : \Omega \times \mathcal{F} \to [0,1]$ be a map such that for every $\omega \in \Omega$, $\mu(\omega, \cdot)$ is a probability measure on $(\Omega, \mathcal{F})$ and for every $A \in \mathcal{F}$

$$\mu(\omega, A) = \mathbb{P}(A \mid \mathcal{G})(\omega), \quad \text{a.s.}$$

Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}|X| < \infty$. Show by giving all steps that

$$\mathbb{E}(X \mid \mathcal{G})(\omega) = \int_\Omega X(\omega')\mu(\omega, d\omega') \text{ for } \mathbb{P} \text{ a.e. } \omega.$$

2. (*20 points*) Let $X_1, X_2, \ldots$ be real-valued measurable functions on $(\Omega, \mathcal{F})$. Let P and Q be two probability measures on $(\Omega, \mathcal{F})$. Suppose that for each $n \geq 1$, under P, $(X_1, \ldots X_n)$ has a joint probability density function(p.d.f.) $p_n : \mathbb{R}^n \to \mathbb{R}_+$ while under Q the joint p.d.f. is $q_n : \mathbb{R}^n \to \mathbb{R}_+$. Define

$$Y_n = \begin{cases} \frac{q_n(X_1,\ldots X_n)}{p_n(X_1,\ldots X_n)} & \text{if the denominator is non-zero} \\ \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$. Show that $(Y_n, \mathcal{F}_n)$ is a supermartingale and it is a martingale if $Q|_{\mathcal{F}_n}$ is absolutely continuous with respect to $P|_{\mathcal{F}_n}$.

3. (*20 points*) Let $\{X_n\}_{n\geq 1}$ be a uniformly integrable sequence of real random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{\mathcal{G}_n\}$ be a sequence of sub σ-fields of $\mathcal{F}$. Show that the sequence $\{\mathbb{E}(X_n \mid \mathcal{G}_n), n \geq 1\}$ is uniformly integrable.

4. (*15 points*)
Let $\{P_n\}_{n\geq 1}$, P be probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Suppose for every continuous function $f : \mathbb{R}^d \to \mathbb{R}$ with compact support (i.e. the function is zero outside a compact subset of $\mathbb{R}^d$), $\int f dP_n \to \int f dP$ as $n \to \infty$. Show that P_n converges weakly to P.

5. (*15 points*) Let P and Q be probability measures on $(\Omega, \mathcal{F})$. Let $\{X_n\}$ be a sequence of real random variables on $(\Omega, \mathcal{F})$ such that it is stationary and ergodic under both P and Q. Let $P_{X_1} = P \circ X_1^{-1}$ and $Q_{X_1} = Q \circ X_1^{-1}$. Show that either $P_{X_1} = Q_{X_1}$ or they are singular (i.e. they are supported on disjoint sets).

Hint: If $P_{X_1} \neq Q_{X_1}$, for some Borel subset B of $\mathbb{R}$, $P(X_1 \in B) \neq Q(X_1 \in B)$. Now apply the ergodic theorem to $\frac{1}{n}\sum_{i=1}^n 1_{\{X_i \in B\}}$.

6. (*15 points*) Let $X_1, X_2, \ldots$ be independent with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$, $n \geq 1$. Show that (i) $X_n \to 0$ in probability iff $p_n \to 0$ and (ii) $X_n \to 0$ a.s. iff $\Sigma p_n < \infty$.