## STOR641 - Comprehensive Written Exam - August 2014

**This test consists of three questions. The relative weights are specified following each part. You are NOT allowed to refer to any notes, books etc.**

**Good luck!**

**1**− Consider a single-server system at which jobs arrive (and join a queue) according to a Poisson process with rate $\lambda$. Each job is of type-1 with probability $p_1 > 0$, and of type-2 with probability $p_2 = 1 - p_1$ independently of other jobs. The server cannot observe the type of each job, however, it performs a diagnostic task on each job to determine its type. The server operates according to the following rules: (i) As soon as a type-1 job is identified, the job is served right away, (ii) if a type-2 job is identified, the job is placed in a queue for type-2 jobs only, (iii) type-2 jobs are served only when there are no other jobs in the system, (iv) when there are unclassified jobs in the system and there are no jobs identified as type-1, the server performs a diagnostic task on one of the unclassified jobs. The system capacity is two so that whenever there are two jobs in total in the system, arriving jobs are sent away. Each diagnostic task takes an exponential amount of time with rate $\theta > 0$ while service of each type $i$ job (where $i = 1, 2$) takes an exponential amount of time with rate $\mu_i > 0$. All service and diagnosis times are independent of each other. They are also non-premptive meaning that once a service or a diagnostic task starts, the server has to finish the service or diagnosis before moving onto a new job.

- **(a)** Model this system as a continuous-time Markov chain (CTMC). Clearly describe the states and give the transition rate diagram. **(10 points)**
- **(b)** Is the CTMC irreducible? Why or why not? **(5 points)**
- **(c)** Write the balance equations. DO NOT SOLVE THEM. **(5 points)**
- **(d)** Consider a job which has just arrived at an empty system. What is the expected time until this job leaves the system? **(10 points)**
- **(e)** Letting $\pi_i$ denote the steady-state probability that the system is in state $i$, give an expression for the fraction of jobs which are denied access to the system because of capacity. You are NOT expected to solve the balance equations. **(10 points)**
- **(f)** Letting $\pi_i$ denote the steady-state probability that the system is in state $i$, give an expression for the long-run average rate with which type-2 jobs leave the system after receiving service. You are NOT expected to solve the balance equations. **(10 points)**

**2**− Let $\{X_n, n \geq 0\}$ be the success runs Markov chain on $\{0, 1, 2, \ldots\}$ with $P_{i,0} = q > 0$, $P_{i,i+1} = p > 0$ for $i \geq 0$ where $p + q = 1$. Prove or disprove the following statement: $\{X_n, n \geq 0\}$ is positive recurrent. **(26 points)**

**3**− For each one of the following statements, indicate whether the statement is true or false and explain your reasoning shortly.

- **(a)** If a DTMC has a finite state space and all of its states communicate with each other then it has a limiting distribution. **(12 points)**
- **(b)** If a DTMC with an infinite state space has a limiting distribution then all of its states must communicate with each other. **(12 points)**