STOR 642
Comprehensive Written Examination
9:00am-1:00pm, August 14, 2014

This test consists of three questions.
This is a closed book exam. No communication devices of any kind are allowed.
Explain your answers in detail.
The problem and subproblem weights are given in parentheses.

Problem 1. (30 points) A clocked queue with clock cycle $d > 0$ operates as follows: Customers arrive at this queue according a PP(λ) and form a single line. At time nd ($n = 0, 1, 2, 3, \cdots$), the customer at the head of the line (if the line is not empty) is removed from the queue. Let $X(t)$ be the number of customers in the system at time t and $X_n = X(nd^-)$, the number of customers just before time nd, $n \geq 0$.

1. (6) Show that $\{X_n, n \geq 0\}$ is a DTMC and display its transition probability matrix. What is the condition of stability?
2. (4) Is $\{X(t), t \geq 0\}$ a queue length process in an M/G/1 queue? Why or why not?
3. (8) Compute $\lim_{n\to\infty} P(X_n = 0)$ assuming stability.
4. (12) Compute $\lim_{t\to\infty} P(X(t) = 0)$ assuming stability.

Problem 2. (40 points) A machine is subject to shocks that arrive one at a time according to a renewal process with inter-renewal time cdf F. If the machine is working when a shock occurs it fails with probability $\alpha \in (0, 1)$, independent of history. When the machine fails, it undergoes repairs; the repairtimes being iid Exp(λ) random variables. The shocks have no effect on the machine under repair. The repaired machine is as good as new. Let $X(t)$ be the number of working machines at time t. Assume that a shock has occurred at time zero and $X(0+) = 0$.

1. (2) Is $\{X(t), t \geq 0\}$ a CTMC for all distributions F? Why or why not?
2. (2) Is $\{X(t), t \geq 0\}$ an SMP? Why or why not?
3. (2) Is $\{X(t), t \geq 0\}$ an ARP? Why or why not?
4. (2) Is $\{X(t), t \geq 0\}$ a regenerative process? Why or why not?
5. (2) Let S_n be the time of the nth shock. Is S_n a regenerative epoch for the process $\{X(t), t \geq 0\}$ for every $n \geq 0$?
6. (5) Let $X_n = X(S_n^+)$, the state of the machine just after the nth shock. Show that $\{(X_n, S_n), n \geq 0\}$ is a Markov renewal sequence. Compute its kernel.

7. (5) Show that $\{X(t), t \geq 0\}$ is an MRGP with the Markov renewal sequence described in part 6.

8. (10) Let
$$H_{ij}(t) = P(X(t) = j | X(0) = i), \quad i, j = 0, 1, 2, t \geq 0.$$
Show that H satisfies the Markov-renewal equation
$$H(t) = D(t) + G * H(t).$$
Write down the matrix $D(t)$ explicitly.

9. (10) Compute the limiting distribution of $\{X(t), t \geq 0\}$.

Problem 3. (30 points) Let $\{X(t), t \geq 0\}$ be a BM(μ, σ), with $\mu > 0$. Let $a > 0$ and define
$$T_a = \min\{t \geq 0 : X(t) = a\}.$$

1. (10) Show that
$$m(x) = E(T_a | X(0) = x), \quad x < a.$$
satisfies the differential equation
$$\frac{\sigma^2}{2} m''(x) + \mu m'(x) = -1.$$
What are the boundary conditions?

2. (10) Solve the equation and show that
$$E(T_a | X(0) = 0) = \frac{a}{\mu}.$$

3. (10) For $i = 1, 2$, let $\{Y_i(t), t \geq 0\}$ be a BM(μ_i, σ_i), with $\mu_1 > \mu_2$. Suppose the Y_1 process is independent of the Y_2 process and $Y_1(0) = y_1 < Y_2(0) = y_2$, where y_1 and y_2 are given real numbers. Define
$$T = \min\{t \geq 0 : Y_1(t) = Y_2(t)\}.$$
Compute $E(T)$.