

**STOR 762**  
**Comprehensive Written Exam**  
**August 13, 2014**

- This exam consists of 3 questions on 2 pages.
- The exam is closed book and closed notes.
- You are NOT allowed to use a calculator or a cell phone throughout the exam.
- Explain your answers in detail.

**Problem 1.** (30 points) Let  $f(x) = \sqrt{2/\pi}e^{-x^2/2}$ ,  $x \geq 0$ , be the density of the absolute value  $X = |Z|$  of a standard normal random variable  $Z$ .

- (a) Provide an acceptance-rejection algorithm to generate random variates from  $f(x)$  using the majorizing function  $t(x) = \sqrt{2e/\pi}e^{-x}$ .
- (b) The density of a standard normal distribution is given by

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}, \text{ for } -\infty < z < \infty.$$

Devise a composition algorithm to generate a standard normal variate  $Z$  based on the distribution for  $X = |Z|$ .

- (b) What is the expected number of random numbers used to generate a single variate using the algorithm provided in part (b)?

**Problem 2.** (40 points) Suppose that  $\{X_i, i \geq 1\}$  is an independent and identically distributed sequence of random variables with a finite mean  $\mu$ . Define a new stochastic process  $\{Y_i, i \geq 1\}$ , where  $Y_i = X_i + a^i$  and  $a$  is a constant between zero and one.

(a) Consider a single replication of the simulated process  $\{Y_i, i = 1, \dots, n\}$ , where  $n$  is the replication length. Obtain a point estimator for the steady-state mean of the process  $\{Y_i\}$  using all  $n$  observations. Provide an expression for the expected value of this point estimator in terms of  $\mu$ ,  $a$ , and  $n$ .

(b) Suppose now that  $k$  independent replications of the same process are conducted with each replication having  $n/k$  observations. (Assume that  $n/k$  is a positive integer.) The observations from the  $j$ th replication are given by  $\{Y_{j,i}, i = 1, \dots, n/k\}$  for  $j = 1, \dots, k$ . Obtain a point estimator for the steady-state mean of the process  $\{Y_i\}$  using all  $k$  replications. Provide an expression for the expected value of this point estimator in terms of  $\mu$ ,  $a$ ,  $k$ , and  $n$ .

(c) Obtain the absolute bias of each of the estimators found in parts (a) and (b) and find which one is smaller.

(d) What is the covariance between any two observations that come from the same replication of the process  $\{Y_i\}$ ?

(e) Construct a  $(1 - \alpha)100\%$  confidence interval on the steady-state mean of the process  $\{Y_i\}$  using the observations from part (a).

(f) Construct a  $(1 - \alpha)100\%$  confidence interval on the steady-state mean of the process  $\{Y_i\}$  using the observations from part (b). Which confidence interval will have a larger coverage, this one or the one obtained in part (e)? Discuss your answer.

**Problem 3.** (30 points) Consider the Gamma( $\alpha, \beta$ ) distribution with p.d.f.

$$f(x) = \frac{\beta^{-\alpha} x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)}, \text{ for } x > 0,$$

mean  $\alpha\beta$ , and variance  $\alpha\beta^2$ , where  $\Gamma(\alpha)$  is the gamma function.

- (a) Given a sample  $x_1, x_2, \dots, x_n$  of independent and identically distributed random values from a Gamma( $\alpha, \beta$ ) distribution with a *known* shape parameter  $\alpha$ , find the maximum likelihood estimator  $\hat{\beta}$  of  $\beta$ .
- (b) Is the maximum likelihood estimator obtained in part (a) an unbiased estimator for  $\beta$ ? Why or why not?
- (c) Find the maximum likelihood estimator for the variance of a gamma random variable with a known shape parameter  $\alpha$ .