DEPARTMENT OF STATISTICS AND OPERATIONS RESEARCH
OPERATIONS RESEARCH DETERMINISTIC QUALIFYING EXAMINATION

August 11, 2014
9:00 am - 1 pm

General Instructions

This examination is closed-book, and consists of three questions. Answer all three as clearly and concisely as you are able. Use of the internet and/or mobile devices is not permitted.

Question 1. (33 points)

a. (11 points) For all positive integers n describe a directed graph with n nodes, which has at least 2^{cn} distinct directed cycles, for some fixed $c > 0$.

   (There is no need to optimize c; any $c > 0$ will do.)

b. (22 points) You are given a directed graph $G = (N, A)$ with n nodes and m arcs.

   Describe an integer programming problem which determines the minimum number of arcs that must be removed from G so the remaining graph has no directed cycle.

   The IP should have $O(n + m)$ variables and constraints; the range of all variables (i.e. the difference between their upper and lower bounds) should also be $O(n + m)$.

   Carefully describe in words the meaning of all variables and constraints.

   (Hint: a directed graph has no directed cycle, if and only if it has a . . .)

Question 2. (33 points) Suppose that $B \in \mathbb{R}^{m\times m}$ is a positive semi-definite matrix (i.e., $x^T Bx \geq 0$ for all $x \in \mathbb{R}^m$). Let $b \in \mathbb{R}^m$ be a given vector. Consider the set

$$S = \{y \in \mathbb{R}^m \mid By \geq b,\ y \geq 0\}.$$

Suppose that S is nonempty. Prove that S is an unbounded set.

Hint: A subset S of an Euclidean space is said to be bounded, if there exists a positive real number M such that every $x \in S$ satisfies $\|x\|_2 \leq M$. We say S is unbounded, if it is not bounded. You can use the fact that S is unbounded if and only if there exists a vector $c \in \mathbb{R}^m$ such that the LP $\max_{y\in S} c^T y$ is an unbounded LP.

Question 3. (34 points) Consider the following linear program in canonical form, given in a simplex tableau.

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs	basic var
1	4	-3	-2	0	0	0	10	$z = 10$
0	0.6	-0.5	-0.4	1	0	0	2	$x_4 = 2$
0	-0.2	0.8	-0.5	0	1	0	3	$x_5 = 3$
0	-0.3	-0.2	1	0	0	1	4	$x_6 = 4$

max z; $x \geq 0$

Prove the following statements.

a. (12 points) In any feasible solution x of the above linear program, at least one of the two components x_1 and x_4 is strictly positive.

b. (22 points) In any basic feasible solution x of the above linear program, one of the two components x_1 and x_4 is exactly zero. (Hint: look at other pairs of variables as well.)