DEPARTMENT OF STATISTICS AND OPERATIONS RESEARCH
THE UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL
———————— AUGUST, 2020 ————————

OPTIMIZATION QUALIFYING EXAMINATION

Information:

- **Student's full name:** ______________________________
- **Student's signature:** ____________________
- **Date:** August 3, 2020. **Time:** 9:00 AM - 1:00 PM.
- **Honor pledge:** I have neither given nor received unauthorized assistance on this exam.

General instructions: Please carefully read the following instructions before answering.

a. Write your name on this sheet. Date and sign on the first page.
b. This examination is closed-books and notes, and consists of four questions.
c. Answer all four questions as clearly and concisely as you are able.
d. Use of the internet, computers, and/or mobile devices is not permitted.

The exam questions

Question 1: Consider the black-and-white traveling salesman problem (TSP) as follows: We are given a directed graph $G = (N, A)$ with $n = |N|$ and we assume that n is even. We seek a tour, i.e. a directed cycle that visits all nodes. For each arc $(i, j) \in A$ we are given two real numbers b_{ij} and w_{ij}.

Our goal is to find a tour and color each of its arcs black or white (not both). The black and white arcs should alternate, as black, white, black, white, $\ldots$, black, white. If an arc (i, j) is colored black, then its cost is b_{ij}. If an arc is colored white, then its cost is w_{ij}. The cost of the tour should be minimized.

Formulate this problem as a mixed integer program (MIP). Carefully explain the meaning of the variables and constraints.

Question 2: An undirected graph $G = (V, E)$ is called bipartite, if V can be partitioned into subsets V_1 and V_2 such that all edges are between V_1 and V_2, meaning $(i, j) \in E$ implies $i \in V_1$, $j \in V_2$ or $i \in V_2$, $j \in V_1$.

We are given an undirected graph $G = (V, E)$. We want to remove the minimum number of edges to make G bipartite.

Describe an MIP to accomplish this problem. Carefully explain the meaning of the variables and constraints.

Question 3: Consider the following parametric linear program:

$$\begin{cases} \max_{x\in\mathbb{R}^5} & z(\alpha,\beta) := \alpha x_1 + \beta(x_2 + x_3 - x_4) \\ \text{s.t.} & x_1 + 3x_2 - 2x_3 + x_4 = 5 \\ & x_1 + 4x_2 - 2x_3 + x_4 + x_5 = 9 \\ & -x_1 + 2x_2 + x_4 + x_5 = 7 \\ & x \geq 0, \end{cases} \tag{LP}$$

where $\alpha, \beta \in \mathbb{R}$ are two given parameters. Let us denote by $\mathcal{F}$ the feasible set of (LP), i.e., the set of all vectors $x \in \mathbb{R}^5$ that satisfy all the constraints of (LP). In the following questions, do not use graphical method.

(a) Find a basic feasible solution (BFS) $\hat{x}$ of (LP) knowing that $\hat{x}_2 = 1$ and $\hat{x}_5 = 3$.

(b) Is the feasible set $\mathcal{F}$ of (LP) bounded? Mathematically justify your answer.

(c) Find all α and β such that the BFS $\hat{x}$ found in part (a) is an optimal solution of (LP).

Question 4: Consider the following regularized least-squares problem:

$$\min_{\beta\in\mathbb{R}^p} \left\{ \frac{1}{2}\|X^\top\beta - y\|_2^2 + \rho\|\beta\|_1 + \frac{\gamma}{2}\|\beta\|_2^2 \right\}, \tag{CP}$$

where $\beta \in \mathbb{R}^p$ is a vector of optimization variables, $X \in \mathbb{R}^{p\times n}$ and $y \in \mathbb{R}^n$ are input data, and $\rho > 0$ and $\gamma \geq 0$ are given regularization parameters. Here, $\|u\|_2 := \sqrt{\sum_j u_j^2}$ is the Euclidean norm of a vector u, and $\|u\|_1 := \sum_j |u_j|$ is the ℓ_1-norm of u.

Let us define

$$f(\beta) := \frac{1}{2}\|X^\top\beta - y\|_2^2, \quad g(\beta) := \rho\|\beta\|_1 + \frac{\gamma}{2}\|\beta\|_2^2, \quad \text{and} \quad \Phi(\beta) := f(\beta) + g(\beta).$$

(a) Compute the proximal operator $\mathrm{prox}_{\eta g}(\beta)$ of $\eta g(\cdot)$ for any scalar $\eta > 0$, i.e.:

$$\mathrm{prox}_{\eta g}(\beta) := \arg\min_{\hat{\beta}\in\mathbb{R}^p} \left\{ g(\hat{\beta}) + \tfrac{1}{2\eta}\|\hat{\beta} - \beta\|_2^2 \right\}.$$

Provide the details of your derivation.

(b) Assume that we apply an accelerated proximal gradient (APG) algorithm to solve (CP) with the following input data:

$$X = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 1 & 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad y = (1, -2, 3)^\top.$$

Let $\beta^\star = (1, 0)^T$ be an optimal solution of (CP) in this case. We also use $\beta^0 = (0, 0)^\top$ as an initial point. Using the convergence theorem of APG, estimate the maximum number of iterations so that APG can achieve an approximate solution β^k of (CP) such that $\Phi(\beta^k) - \Phi(\beta^\star) \leq 10^{-4}$. Provide the details of your derivation.

The end