STOR 634 CWE (2020)

Instructions:
There are 5 problems. Each problems is worth 10 points.
Show your work and put down your attempt even if it is not a complete solution.
Partial credit will be assigned so try your best on each question. Good luck!

P1. Let μ^* be an outer measure on Ω and $E \subset \Omega$ such that $\mu^*(E) + \mu^*(E^c) = \mu^*(\Omega) < \infty$.

a) Recall that for all $\epsilon > 0$ there exists a μ^*-measurable $A \supset E$ with $\mu^*(A) \leq \mu^*(E) + \epsilon$. Show that $\mu^*(E^c \cap A) \leq \epsilon$.

b) Prove that $\mu^*(B \cap E) + \mu^*(B \cap E^c) \leq \mu^*(B)$ for all $B \subset \Omega$.

c) Conclude that is E is μ^*-measurable.

P2. Let μ be a measure on $\mathcal{B}(\mathbb{R})$ that is finite for bounded sets and is translation invariant, i.e.,

$$\mu(A + x) = \mu(A) \quad \text{for all } x \in \mathbb{R} \text{ and } A \in \mathcal{B}(\mathbb{R}).$$

Show that $\mu(A) = c\lambda(A)$ where $c = \mu([0, 1]) \geq 0$. (Here, λ is the Lebesgue measure.)

P3. Let $\{Z_n\}$ be i.i.d. standard normal random variables and let $\{a_n\}$ be a sequence of nonnegative real numbers. Prove that $\sum_{n=1}^{\infty} a_n Z_n^2 < \infty$ almost surely if and only if $\sum_{n=1}^{\infty} a_n < \infty$.

P4. For distribution functions F, G on the real line, define the Kolmogorov metric as

$$K(F, G) = \sup_{x \in \mathbb{R}} |F(x) - G(x)|$$

a) Show that if $K(F_n, F) \to 0$ as $n \to \infty$ then F_n converges in distribution to F.

b) Prove that if F_n converges in distribution to F, and F is continuous, then $K(F_n, F) \to 0$.

P5. Suppose that X_1, X_2 and X are independent, identically distributed random variables with mean 0 and variance 1. Prove that

$$\frac{X_1 + X_2}{\sqrt{2}} \stackrel{d}{=} X$$

if and only if all random variables are standard normal.