STOR 635 CWE (2020)

Read the following information before starting the exam: Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer. There are 5 problems. Good luck!

1. (10 points) Suppose that $\{X_n\}$ is a nonnegative $\mathcal{F}_n$-supermartingale and τ is a stopping time. Show that $E(X_\tau) \le E(X_0)$.

2. At time $n = 0$, an urn contains one blue ball and one red ball. At each time $n = 1, 2, \ldots$, a ball is chosen uniformly at random from the urn and then replaced along with an additional ball of the same color. Just after time n, there are $n + 2$ balls in the urn, and denote the number of blue and red balls by B_n and R_n respectively. Define

$$M_n := \frac{B_n}{n+2}.$$

(i) (5 points) Show that $\{M_n : n \ge 0\}$ is a martingale with respect to a natural filtration that you should specify.

(ii) (5 points) Show that

$$\mathbb{P}\left(B_n = k\right) = \frac{1}{n+1}, \ 1 \le k \le n+1.$$

(Hint: Recall that the draws are exchangeable.)

(iii) (5 points) Does $\lim_{n\to\infty} M_n$ exist? What is its distribution? You should show all calculations. Just writing the distribution will not suffice. You can use the result of part (ii) above even if you haven't shown it.

(iv) (5 points) Let τ be the first time a blue ball is chosen. Show that

$$\mathbb{E}\left(\frac{1}{\tau+2}\right) = \frac{1}{4}.$$

Justify all steps.

3. (10 points) Let $\{X_n\}_{n\ge 0}$ be a submartingale with $\sup X_n < \infty$ a.s. Let $\xi_n = X_n - X_{n-1}$ and suppose $\mathbb{E}(\sup_n \xi_n^+) < \infty$. Show that X_n converges a.s.

4. (i) (5 points) Show that a collection $\mathcal{H} \in L^1(\mu)$ of functions is uniformly integrable if and

only if there is $H : \mathbb{R}_+ \to \mathbb{R}_+$ such that $H(x)/x \to \infty$ as $x \to \infty$ and $\sup_{f \in \mathcal{H}} \int H(|f|) d\mu < \infty$.

(ii) (5 points) Suppose $\mathcal{H} \in L^1(\mu)$ is a uniformly integrable family of functions. Show that

$$\operatorname{conv} \mathcal{H} := \left\{ \sum_{i=1}^n \alpha_i f_i \ : \ n \in \mathbb{N},\ f_i \in \mathcal{H},\ \alpha_i \geq 0,\ \sum_{i=1}^n \alpha_i = 1 \right\}$$

is a uniformly integrable family.

5. (10 points) Let S_n be the Markov chain on $\mathbb{Z}$ with transition probabilities

$$p(x, x+1) = p(x, x-1) = 1/2,\ \forall\ x \in \mathbb{Z}.$$

Denote by $\mathbb{E}_i(\cdot)$ and $\mathbb{P}_i(\cdot)$ the expectation and probability under the law of the chain started from $i \in \mathbb{Z}$. Let

$$\tau := \inf\{n \geq 0 : |S_n| = 10\}.$$

Show that there exists $\theta > 0$ such that $\mathbb{E}_0\left(e^{\theta\tau}\right) < \infty$.

(Hint: Obtain upper bound of $\sup_{i \in \mathbb{Z}:|i| \leq 10} \mathbb{P}_i\left(\tau > 20k\right)$ in terms of $k \geq 1$ and use it and Markov property to compute the exponential moment above.)