STOR641 - Comprehensive Written Exam - August 2020

This test consists of three questions. The relative weights are specified following each part. You are NOT allowed to refer to any notes, books etc.

Good luck!

1− State for each statement below whether or not it is true. If your answer is "true" then either give a proof or provide a rigorous justification for your answer; if your answer is "false" then provide a counterexample.

(a) If a discrete-time Markov chain (DTMC) has a finite number of states and is irreducible then it has a limiting distribution.**(7 points)**

(b) If a continuous-time Markov chain (CTMC) has a finite number of states and is irreducible then it has a limiting distribution.**(8 points)**

(c) Suppose that i and j are two communicating states in a CTMC, which is NOT irreducible, and $\lim_{t\to\infty} P_{ij}(t) = 0$ where $P_{ij}(t)$ is the transition probability from state i to j. Then, i and j cannot be in a closed communicating class. **(7 points)**

(d) It is not possible for a CTMC which has a limiting distribution to have an embedded DTMC, which does not have a limiting distribution. **(8 points)**

2− A service facility has decided to serve its customers at two separate stations depending on whether the customers are wearing face masks or not. At each station, there is a single server which serves the customers in a First-Come-First-Served fashion. Customers who wear masks arrive according to a Poisson process with rate λ_f (where $0 < \lambda_f < \infty$) and are served in Station 1 while customers who do not wear masks arrive according to an independent Poisson process with rate λ_n (where $0 < \lambda_n < \infty$) and are served in Station 2. In Station 1, service times are i.i.d. with an exponential distribution with parameter μ_f (where $0 < \mu_f < \infty$) and in this station as soon as a customer's service is over the next customer in line can start receiving service. In Station 2, service times are i.i.d. with an exponential distribution with parameter μ_n (where $0 < \mu_n < \infty$). Unlike in Station 1, in Station 2, in between two customer services, the equipment needs to be disinfected. Therefore, once the service of a customer is over, the server becomes unavailable for an independent and exponentially distributed time with parameter θ (where $0 < \theta < \infty$).

(a) Suppose that Station 1 is empty and in Station 2 there is only one customer and that customer is waiting for the disinfection process to be completed. What is the probability that the next customer to depart from the service facility will be from Station 1? **(10 points)**

(b) Suppose that Station 2 is empty. What is the expected time until the second departure from Station 2? **(10 points)**

(c) Suppose now that the two stations are merged so that all customers are served in a single station with two servers working in parallel. Each server can now serve both types of customers but similar to the earlier setup, a server who serves a customer who does not wear a face mask will need to be disinfected before serving another customer. Assume that currently both servers are busy serving customers and a customer has just arrived. What is the expected time until this customer starts receiving service? **(10 points)**

3– Consider the following simplified model for disease spread. There are N individuals in a town. Each individual is either susceptible for infection (denoted by S), currently infected (denoted by I), or has recently recovered from the infection (denoted by R). We assume that if an infected individual dies then a new individual is added to the population (as part of the susceptible group) right away so that the total number of individuals in the town is N at all times. Any one of the susceptible individuals can get infected on any given day. We assume that on any given day t this happens independently for each individual with probability $p(t) = aX_I(t)/N$ where a is a constant such that $a \leq 0 \leq 1$ and $X_I(t)$ is the number of infected individuals on day t. An infected individual can either die (with probability q, where $0 < q < 1$, independently of everything else) or recover from the disease (with probability $1 - q$). For individuals who die, the number of days between getting infected and death are i.i.d. random variables with a Geometric distribution with parameter α, where $0 < \alpha < 1$. For individuals who recover, the number of days between getting infected and recovery are i.i.d. random variables with a Geometric distribution with parameter β, where $0 < \beta < 1$. Individuals who recover are immune from the infection for a period of time at the end of which they become susceptible again. This time is assumed to be i.i.d. with a Geometric distribution with parameter γ, where $0 < \gamma < 1$. Assume that on each day first new infections occur followed by recovery of the infected people and finally deaths and transition of the recovered individuals whose recovered time is over to the susceptible stage.

(a) Model this system as a discrete-time Markov chain. Clearly describe the state space $\mathcal{S}$ and give expressions for the transition probabilities. Note that writing down the transition probabilities exactly might take some time. As long as your state representation is correct and you are on the right track when writing the transition probabilities you will get full credit. **(20 points)**

(b) For any $j \in \mathcal{S}$, let π_j denote the steady-state probability that the system is in state j. Give an expression for the long-run fraction of time there are more than K individuals who are infected. Give your answer using π_js. DO NOT SOLVE THE BALANCE EQUATIONS. **(10 points)**

(c) For any $j \in \mathcal{S}$, let π_j denote the steady-state probability that the system is in state j. Give an expression for the long-run number of deaths per day. Give your answer using π_js. DO NOT SOLVE THE BALANCE EQUATIONS. **(10 points)**