STOR 642
Comprehensive Written Examination
9:00am-1:00pm, August 6, 2020

This test consists of two questions.
This is a closed book exam. No communication devices of any kind are allowed.
Explain your answers in detail.
The problem and subproblem weights are given in parentheses.
Contact me at 919 491 9408 by phone or text if you have any questions.

Problem 1. (50 points)

1. (5) Define a renewal process $\{N(t), t \geq 0\}$ generated by $\{X_n, n \geq 1\}$.

2. (5) Define recurrence and transience of a renewal process. Give a necessary and sufficient condition of recurrence.

3. (5) Define periodicity (aperiodicity) of a recurrent renewal process.

   For the remaining subparts of this question assume that $\{N(t), t \geq 0\}$ is an aperiodic, recurrent renewal process with inter-renewal times with common cdf F, mean $\tau < \infty$ and variance σ^2. Define

$$Z(t) = \begin{cases} 0 & \text{if } N(t) \text{ is even} \\ 1 & \text{if } N(t) \text{ is odd.} \end{cases}$$

4. (5) Draw a sample path of $\{Z(t), t \geq 0\}$. Derive a renewal type equation satisfied by

$$H(t) = P(Z(t) = 0), \quad t \geq 0.$$

5. (5) Show that the key renewal theorem is applicable to the renewal type equation derived above.

6. (5) Using the key renewal theorem, compute

$$c = \lim_{t \to \infty} H(t).$$

7. (5) Show that $\{Z(t), t \geq 0\}$ is an alternating renewal process (ARP). Compute the limit c defined above using the theory of the limiting behavior of an ARP.

8. (5) Show that $\{Z(t), t \geq 0\}$ is a semi-Markov process (SMP). Compute the limit c defined above using the theory of the limiting behavior of an SMP.

9. (5) Define

$$R(t) = \int_0^t Z(u)du, \quad t \geq 0.$$

Draw a typical sample path of $\{R(t), t \geq 0\}$. Show that $\{R(t), t \geq 0\}$ is a renewal reward process. Compute

$$\lim_{t\to\infty} \frac{R(t)}{t}.$$

10. (5) Let S_n be the nth renewal epoch in the renewal process, and define

$$W(t) = \int_{S_{2n}}^t Z(u)du, \quad S_{2n} \leq t < S_{2n+2}, \quad n \geq 0.$$

Draw a typical sample path of $\{W(t), t \geq 0\}$. Show that $\{W(t), t \geq 0\}$ is a regenerative process. Compute

$$H(x) = \lim_{t\to\infty} P(W(t) \leq x).$$

Problem 2. (50 Points.) Consider a single server queue with iid $\exp(\mu)$ service times and $PP(\lambda)$ arrivals. Suppose each customer in the system (including any in service) costs h dollars per unit time. It costs g dollars per unit time to keep the server idle. Our aim is to design an admission control policy that minimizes the expected total cost of operating the system. We consider three controls.

Static Control. The static admission control is described by a single parameter $p \in [0, 1]$: each arrival is admitted with probability p and rejected with probability $1 - p$. Admitted customers join the queue and wait until they get served. Rejected customers go away.

1. (5) Compute $C(p)$, the long run expected holding + idleness cost per unit time.
2. (5) Compute the optimal value p^* of p that minimizes this cost. This defines policy $A(p^*)$.

Discounted Cost Dynamic Control. The dynamic control (also called admission policy) allows us to admit or reject an arriving customer based on the number customers in the system at the time of their arrival. We want to find an admission policy that minimizes the expected total discounted cost over infinite horizon, with continuous discount factor $\alpha > 0$.

3. (5) Formulate the dynamic admission control problem as a Markov Decision Process.
4. (10) Derive the optimality equation satisfied by the optimal value function v.
5. (5) Show how one can compute v it by value iteration.
6. (10) Suppose it is possible to prove that v is a convex function of i. Show that this implies that there is an integer k^* such that it is optimal to admit an arriving customer if the number in the system is less than k^*, and reject the customer otherwise. This defines policy $\Pi(k^*)$.

Average Cost Dynamic Control. Let m be a given positive integer. Let $\Phi(m)$ be the admission policy that admits an arriving customer if there are less than m customers in the system, and rejects them otherwise.

7. (5) Compute $D(m)$, the expected long run holding + idleness cost per unit time, under policy $\Phi(m)$. One can find an optimal m^* numerically that minimizes this cost.
8. (5) Which policy do you think will be better in terms of the long run holding + idleness cost per unit time: $A(p^*)$ or $\Phi(m^*)$? Give an intuitive argument.