STOR 641
Comprehensive Written Exam
Thursday, August 18, 2011

Write your assigned letter on each sheet of your answer book. Do not write your name anywhere.
This test consists of three problems.
All problems carry equal weight.
This is a closed book exam.
Explain your answers in detail.
The duration of the exam is two hours.

Problem 1. A company manufacturing digital cameras introduces a new model every year (and discontinues all the previous models). Its marketing department has developed the following model of consumer behavior. Consumers make camera buying decisions once a year. A consumer who has a k year old camera from this company will, independently of the past, buy a new camera from this company with probability α_k, or keep the current camera with probability β_k, or switch to another company with probability $1 - \alpha_k - \beta_k$, $(0 \le k \le K-1)$ and be lost forever. We assume that $\beta_K = 0$ for a given $K > 0$. That is, a customer with K year old camera will either upgrade to new one from this company, or will be lost to the competition. Let B_n be the number of first time buyers of the camera from this company in year n. (Buyers of a new camera are said to have a zero-year-old camera.) Suppose $\{B_n, n \ge 0\}$ is a sequence of iid non-negative integer-valued random variables with common mean b. Let $X_n(k)$ be the number of customers who have a k year old camera from this company in year n $(0 \le k \le K,\ \ n \ge 0)$. Let $X_n = [X_n(0), X_n(1), \cdots, X_n(K)]$.

(a) Show that $\{X_n, n \ge 0\}$ is a DTMC. What is its state space? Is it irreducible and aperiodic? Assume that $\alpha_k > 0$ for $0 \le k \le K$ and $\beta_k > 0$ for $0 \le k < K$, with $\beta_K = 0$.

(b) Suppose the company enters the market for the first time in year 0. Let $x_n(k) = E(X_n(k))$ and $x_n = [x_n(0), x_n(1), \cdots, x_n(K)]$. Derive a recursive equation giving x_{n+1} in terms of x_n. Compute

$$x_\infty = \lim_{n\to\infty} x_n.$$

Hint: You may find the following matrix notation useful:

$$M = \begin{bmatrix} \alpha_0 & \beta_0 & 0 & \cdots & 0 & 0 \\ \alpha_1 & 0 & \beta_1 & \cdots & 0 & 0 \\ \alpha_2 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{K-1} & 0 & 0 & \cdots & 0 & \beta_{K-1} \\ \alpha_K & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

(c) Let s_n be the expected number of cameras the company sells in year n. Compute s_n in terms of x_n, and its limit as $n \to \infty$.

(d) Suppose the conditions of part (a) hold. Use Foster's criterion given below to show that this DTMC is always positive recurrent.

Foster's Criterion: Let $\{X_n, n \geq 0\}$ be an irreducible DTMC on a countable state space S. If there exists a potential function $\nu : S \to [0, \infty)$, an $\epsilon > 0$, and a finite set $H \subset S$ such that
1. $|E(\nu(X_{n+1}) - \nu(X_n)|X_n = x)| < \infty$ for all $x \in S$,
2. $E(\nu(X_{n+1}) - \nu(X_n)|X_n = x) < -\epsilon$ for all $x \notin H$
the DTMC is positive recurrent.

Hint: You may try $\nu(x) = \sum_{k=0}^{K} x(k)$ and use the result derived in part (b) above.

Problem 2. Customers arrive at a shop according to a Poisson process with rate λ. Assume that there is space for each arriving customer in the shop. Upon arrival each customer spends some time in the shop browsing, which is uniformly distributed over (0,1) independently of the other customers. Then, the customer either leaves immediately without buying anything or joins the checkout queue. The probability that a customer decides to join the checkout queue is proportional to how much time she spends browsing. In particular, if the customer spends x units of time looking around, the probability that she will decide to buy something and join the checkout queue is x.

(a) What is the long-run average arrival rate to the checkout queue?

(b) Let q_t denote the probability that a customer who entered the shop some time during $(0, t)$ joins the checkout queue by time t. Compute q_t.

(c) Let $A(t)$ denote the number of arrivals to the check out queue over $(0, t]$. Determine $E(A(t))$.

(d) Is $\{A(t), t \geq 0\}$ a Poisson Process? Is it a non-homogeneous Poisson Process?

Problem 3. Consider a call center with s servers. The service times at the ith server are iid exponential random variables with parameter μ_i. Without loss of generality, assume that

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s.$$

The customers arrive according to a Poisson Process with rate λ. An arriving customer waits in the queue if all servers are busy, else he goes to the fastest idle sever. When a server completes a service he starts serving a waiting customer, if one is waiting. If there are no waiting customers, he takes over the service of a customer who is being served by the slowest busy server that is slower than him. If there are no waiting customers and there is no busy server that is slower than him, he stays idle, waiting for a new customer. Let $X(t)$ be the number of customers in the system at time t.

(a) Is $\{X(t), t \geq 0\}$ a birth and death process? If so, give its birth and death parameters and draw its rate diagram.

(b) What is the condition of stability? Assuming the system is stable, let

$$p_i = \lim_{t \to \infty} P(X(t) = i), \quad i \geq 0$$

Write the balance equations satisfied by p_i, $i \geq 0$.

(c) Compute the steady state distribution p_i, $i \geq 0$.

(d) What is the probability that server i is busy?

(e) The queueing discipline is obviously unfair to the faster servers. We can make it fair by paying the faster servers more. Suppose we pay server i c_i dollars per hour that he is busy. Find the c_i that will make the long run pay rate the same for all the servers.