STOR/MATH 634: Graduate Probability I

Comprehensive Written Exam

August 2021

FILL OUT THE INFORMATION IN THE BOX

Name:_____

READ THE FOLLOWING INFORMATION.

- **This is a closed book and notes exam.** There are 4 questions. Attempt all questions. You may appeal to any result proved in class without proof unless you are specifically asked to "**Give a complete proof.** ".
- All questions are worth the same total amount (15 pts).
- Breathe and try to relax and think about each problem slowly. Remember: I am testing not just if you can get the final "correct" answer but your technique in approaching problems. Partial credit will be given even if you are not able to completely solve a problem but make headway in solving the problem.

_____/15 **Q1.** Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{\mathcal{F}_k : k \geqslant 1\}$ are a sequence of sub-σ fields of $\mathcal{F}$. Show that the sequence $\{\mathcal{F}_k : k \geqslant 1\}$ is independent **if and only if** each of the pairs $(\sigma(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n), \mathcal{F}_{n+1})$ is independent for $n = 1, 2, \ldots$.

_____/15 **Q2.** Let $\{X_n\}_{n\geqslant 1}$ be a sequence of non-negative random variables all defined on the same space and let $\{m_n\}_{n\geqslant 1}$ be a sequence of strictly positive constants with $m_n \uparrow \infty$. For each fixed $n \geqslant 1$, suppose the probability density function of X_n is given by,

$$f_n(y) = m_n \exp(-m_n y), \qquad y \geqslant 0.$$

(Note: this implies for any $t \geqslant 0$ that $\mathbb{P}(X_n \geqslant t) = \exp(-m_n t)$). Define the random variable

$$Y(\omega) = \sup_{n\geqslant 1} X_n(\omega), \qquad \omega \in \Omega.$$

_____/10 (a) Suppose $m_n = \log(n+1)$ for all $n \geqslant 1$. Show that

$$\mathbb{P}(Y < \infty) = 1.$$

_____/5 (b) Suppose $m_n = \log\log(n+4)$ for $n \geqslant 1$ (the "$n+4$" is just to ensure that $m_n > 0$ and increasing; to understand the problem it is easier to just thinking of the above as the sequence "$\log\log n$"). Further assume $\{X_n\}_{n\geqslant 1}$ are independent collection of random variables. Show that

$$\mathbb{P}(Y = \infty) = 1.$$

_____/15 **Q3.** Suppose $(\Omega, \mathcal{F}, \mu)$ is a general σ-finite measure space and let $\{f_n\}_{n \geqslant 1}$ be measurable collection of functions. Suppose $g \geqslant 0$ is another measurable and integrable function such that $|f_n(\omega)| \leqslant g(\omega)$ for all $\omega \in \Omega$. Consider the function

$$f^*(\omega) = \limsup_{n \to \infty} f_n(\omega), \qquad \omega \in \Omega.$$

Show that $f^*(\omega)$ is integrable and further

$$\int f^* d\mu \geqslant \limsup_{n \to \infty} \int f_n d\mu.$$

_____/15 **Q4.** Fix $\alpha > 0$ and Suppose $\{X_n : n \geqslant 1\}$ is a sequence of independent **Bernoulli** random variables such that for each $n \geqslant 1$, X_n is a Bernoulli$(1/n^\alpha)$ random variable, i.e. for each $n \geqslant 1$,

$$\mathbb{P}(X_n = 1) = \frac{1}{n^\alpha}, \qquad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n^\alpha}.$$

Define $S_n = \sum_{i=1}^{n} X_i$.

_____/5 (a) Show that if $\alpha > 1$ then S_n converges a.s. to a finite random variable.

_____/5 (b) Show that if $\alpha \leqslant 1$ then $S_n \to \infty$ a.s.

_____/5 (c) Show that in the setting of part (4b),

$$\frac{S_n}{\mathbb{E}(S_n)} \xrightarrow{\text{a.e.}} 1$$

Hint: for part(c), this is similar to the final in STOR 634 in Fall 2020. Precisely: Define $n_k = \inf\{n \geq 1 : \mathbb{E}\, S_n \geq k^2\}$. Let $T_k = S_{n_k}$. Show that $T_k / \mathbb{E}(T_k) \to 1$ a.s. by arguing that for every fixed $\delta > 0$

$$\sum_{k=1}^{\infty} \mathbb{P}(|T_k - \mathbb{E}(T_k)| > \delta\, \mathbb{E}\, T_k) < \infty.$$

Try to use this to complete the proof of (c).