STOR 634, CWE 2021-22.

Each problem is 10 points. There are 5 problems in all.

1. (*10 points*) Let $(\Omega, \mathcal{F}, P)$ be a probability space.

a. (*3 pts*) Let $\{X)_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}$-valued random variables on the above probability space. Suppose that $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \varepsilon) < \infty$. Show that $\{\lim_{n\to\infty} X_n$ exists and equals $0\}$ is in $\mathcal{F}$ and that $\mathbb{P}(\lim_{n\to\infty} X_n = 0) = 1$.

b. (*3 pts*) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $\{A_n\}$ be a sequence of events. Show that

$$\mu(\limsup_{n\to\infty} A_n) \geq \limsup_{n\to\infty} \mu(A_n), \quad \mu(\liminf_{n\to\infty} A_n) \leq \liminf_{n\to\infty} \mu(A_n).$$

c. (*4 pts*) Use the previous result to show that if $\{Y_n\}$ is a sequence of real random variables, then, for any $c > 0$,

$$\mu(\limsup_{n\to\infty} Y_n \geq c) \geq \mu(\limsup_{n\to\infty}\{Y_n \geq c\}) \geq \limsup_{n\to\infty} \mu(Y_n \geq c).$$

Give examples to show that in general the two inequalities in the above display cannot be replaced by equalities.

2. (*10 points*) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

a. (*3 pts*) Let $f \in \mathcal{L}^1(\mu)$. Show that $f = 0$ a.e. iff

$$\int_A f d\mu = 0 \text{ for every } A \in \mathcal{F}.$$

b. (*3 pts*) Let $\{h_n\}_{n\geq 1}$ be a sequence of nonnegative measurable functions on the above measure space. Show that

$$\int \left(\sum_{n=1}^{\infty} h_n\right) d\mu = \sum_{n=1}^{\infty} \int h_n d\mu.$$

c. (*4 pts*) Let f be a nonnegative measurable function. For $A \in \mathcal{F}$, let

$$\nu(A) \doteq \int f 1_A d\mu.$$

Show that ν is a measure on $(\Omega, \mathcal{F})$. Show that μ is a finite measure if $f \in \mathcal{L}^1(\mu)$. Finally, show that for any measurable $h : \Omega \to \mathbb{R}$, $h \in \mathcal{L}^1(\nu)$ iff $h \cdot f \in \mathcal{L}^1(\mu)$ in which case $\int h d\nu = \int h f d\mu$.

3. *(10 points)* Let $X_1, X_2, \cdots$ be iid with $0 < X_1 < \infty$ a.s. Let $T_n = X_1 + \cdots + X_n$. Let $N_t = \sup\{n : T_n \leq t\}$. Suppose $\mathbb{E}(X_1) = \mu < \infty$.

a. (*5 pts*) Show that, a.s., for all t

$$\frac{T(N_t)}{N_t} \leq \frac{t}{N_t} \leq \frac{T(N_t+1)}{N_t+1}\frac{N_t+1}{N_t}$$

b. (*5 pts*) Let Ω_0 be the set such that for $\omega \in \Omega_0$, $T_n(\omega)/n \to \mu$ as $n \to \infty$ and $N_t(\omega) \uparrow \infty$. Show that for all such ω, $t/N_t(\omega) \to \mu$.

4. *(10 points)* Let μ_n be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with ch.f. φ_n.

a. (*5 pts*) Suppose that $\{\mu_n\}$ is tight. Show that the sequence $\{\varphi_n\}$ is equicontinuous.

b. (*5 pts*) Suppose next that $\mu_n \to^d \mu_\infty$. Show that $\varphi_n(t) \to \varphi_\infty(t)$ uniformly in $t \in [0, T]$ for every T, where φ_∞ is the ch.f. of μ_∞.

5. *(10 points)*

a. (*5 pts*) Let (Y_1, Y_2) be random variables with joint distribution given as

$Y_2 \backslash Y_1$	1	0
2	0.2	0.4
3	0.1	0.3

Let $\Omega_1 = \{1, 0\}$, $\Omega_2 = \{2, 3\}$, $\mathcal{F}_i = 2^{\Omega_i}$, $i = 1, 2$. Define probability measure μ_1 on $(\Omega_1, \mathcal{F}_1)$ as $\mu_1\{1\} = 0.3$ (values of remaining sets are determined uniquely.) Also define a transition probability function $\mu_2 : \Omega_1 \times \mathcal{F}_2 \to [0, 1]$ as $\mu_2(1, \{2\}) = 2/3$ and $\mu_2(0, \{2\}) = 4/7$ (remaining values are determined uniquely). Let $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$. Let X_1, X_2 be coordinate maps on Ω. Show that (X_1, X_2) has the same distribution as (Y_1, Y_2).

b. (*5 pts*) By invoking Fubini's theorem show that if X is an integrable real random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ then

$$E(X(1 - e^{-|X|})) = \int_0^\infty e^{-t} E[X 1_{|X|>t}] dt.$$