Each problem is 10 points. There are 5 problems in all.

1. (10 points) Let \((\Omega, \mathcal{F}, P)\) be a probability space.

   a. (3 pts) Let \(\{X_n\}_{n \in \mathbb{N}}\) be a sequence of \(\mathbb{R}\)-valued random variables on the above probability space. Suppose that \(\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \varepsilon) < \infty\). Show that \(\lim_{n \to \infty} X_n\) exists and equals 0.

   b. (3 pts) Let \((\Omega, \mathcal{F}, \mu)\) be a measure space. Let \(\{A_n\}\) be a sequence of events. Show that
   \[
   \mu(\limsup_{n \to \infty} A_n) \geq \limsup_{n \to \infty} \mu(A_n), \quad \mu(\liminf_{n \to \infty} A_n) \leq \liminf_{n \to \infty} \mu(A_n).
   \]

   c. (4 pts) Use the previous result to show that if \(\{Y_n\}\) is a sequence of real random variables, then, for any \(c > 0\),
   \[
   \mu(\limsup_{n \to \infty} Y_n \geq c) \geq \mu(\limsup_{n \to \infty} \{Y_n \geq c\}) \geq \limsup_{n \to \infty} \mu(Y_n \geq c).
   \]
   Give examples to show that in general the two inequalities in the above display cannot be replaced by equalities.

2. (10 points) Let \((\Omega, \mathcal{F}, \mu)\) be a measure space.

   a. (3 pts) Let \(f \in L^1(\mu)\). Show that \(f = 0\) a.e. iff
   \[
   \int_A f \, d\mu = 0 \text{ for every } A \in \mathcal{F}.
   \]

   b. (3 pts) Let \(\{h_n\}_{n \geq 1}\) be a sequence of nonnegative measurable functions on the above measure space. Show that
   \[
   \int \left( \sum_{n=1}^{\infty} h_n \right) \, d\mu = \sum_{n=1}^{\infty} \int h_n \, d\mu.
   \]

   c. (4 pts) Let \(f\) be a nonnegative measurable function. For \(A \in \mathcal{F}\), let
   \[
   \nu(A) \doteq \int f 1_A \, d\mu.
   \]
   Show that \(\nu\) is a measure on \((\Omega, \mathcal{F})\). Show that \(\mu\) is a finite measure if \(f \in L^1(\mu)\). Finally, show that for any measurable \(h : \Omega \to \mathbb{R}\), \(h \in L^1(\nu)\) iff \(h \cdot f \in L^1(\mu)\) in which case \(\int h \, d\nu = \int h \cdot f \, d\mu\).
3. (10 points) Let $X_1, X_2, \cdots$ be iid with $0 < X_1 < \infty$ a.s. Let $T_n = X_1 + \cdots + X_n$. Let $N_i = \sup\{n : T_n \leq t\}$. Suppose $\mathbb{E}(X_1) = \mu < \infty$.

a. (5 pts) Show that, a.s., for all $t$
$$\frac{T(N_t)}{N_t} \leq \frac{t}{N_t} \leq \frac{T(N_t + 1)}{N_t + 1}$$

b. (5 pts) Let $\Omega_0$ be the set such that for $\omega \in \Omega_0$, $T_n(\omega)/n \to \mu$ as $n \to \infty$ and $N_t(\omega) \uparrow \infty$. Show that for all such $\omega$, $t/N_t(\omega) \to \mu$.

4. (10 points) Let $\mu_n$ be probability measures on $([0, \infty), \mathcal{B}([0, \infty)))$ with ch.f. $\varphi_n$.

a. (5 pts) Suppose that $\{\mu_n\}$ is tight. Show that the sequence $\{\varphi_n\}$ is equicontinuous.

b. (5 pts) Suppose next that $\mu_n \to^d \mu_\infty$. Show that $\varphi_n(t) \to \varphi_\infty(t)$ uniformly in $t \in [0, T]$ for every $T$, where $\varphi_\infty$ is the ch.f. of $\mu_\infty$.

5. (10 points)

a. (5 pts) Let $(Y_1, Y_2)$ be random variables with joint distribution given as

<table>
<thead>
<tr>
<th>$Y_2 \setminus Y_1$</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.2</td>
<td>0.4</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Let $\Omega_1 = \{1, 0\}$, $\Omega_2 = \{2, 3\}$, $\mathcal{F}_i = 2^{\Omega_i}$, $i = 1, 2$. Define probability measure $\mu_1$ on $(\Omega_1, \mathcal{F}_1)$ as $\mu_1\{1\} = 0.3$ (values of remaining sets are determined uniquely.) Also define a transition probability function $\mu_2 : \Omega_1 \times \mathcal{F}_2 \to [0, 1]$ as $\mu_2(1, \{2\}) = 2/3$ and $\mu_2(0, \{2\}) = 4/7$ (remaining values are determined uniquely). Let $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$. Let $X_1, X_2$ be coordinate maps on $\Omega$. Show that $(X_1, X_2)$ has the same distribution as $(Y_1, Y_2)$.

b. (5 pts) By invoking Fubini’s theorem show that if $X$ is an integrable real random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ then
$$E(X(1 - e^{-|X|})) = \int_0^\infty e^{-t}E[|X|_{|X|>t}]dt.$$