STOR 635, CWE 2020-21

Solutions should be as self contained as possible and should not make reference to theorems, exercises or other results from the class.

1. (*15 points*) Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Let

$$\mathcal{H} = \{g \in L^3(\mu) : \|g\|_3 \leq 1\}.$$

Let

$$\mathcal{F} = \{f : \text{ for some } g_1, g_2 \in \mathcal{H}, f = g_1 g_2\}.$$

Show that $\mathcal{F}$ is a uniformly integrable family.

2. (*15 points*) Let ν, μ be σ-finite measure on $(\Omega, \mathcal{F})$. Suppose that $\mu \ll \nu$ and $\frac{d\mu}{d\nu} > 0$ a.e. ν. Show that $\nu \ll \mu$ and

$$\frac{d\nu}{d\mu} = \left(\frac{d\mu}{d\nu}\right)^{-1} \text{ a.e. } \mu.$$

3. (*15 points*) Suppose that S and T are $\mathcal{F}_n$ stopping times with $S \leq T$. Define:

$$H_n(\omega) = 1_{(S(\omega), T(\omega)]}(n), \; \omega \in \Omega, n \geq 1.$$

Show that $\{H_n\}$ is predictable and deduce that if $\{X_n\}$ is a supermartingale then

$$E(X_{T \wedge n}) \leq E(X_{S \wedge n}) \text{ for all } n.$$

4. (*20 points*) Let $\{X_n\}$ be a $\mathcal{F}_n$- martingale on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for some $M < \infty$, $|X_{n+1} - X_n| \leq M$ for all n. Suppose $X_0 = 0$. Let

$$A \doteq \{\lim X_n \text{ exists and is finite }\}, \; B = \{\limsup X_n = \infty \text{ and } \liminf X_n = -\infty\}.$$

Show $\mathbb{P}(A \cup B) = 1$.

5. (*20 points*) Without referring to any theorems from class show that if a sequence $\{X_i\}_{i \in \mathbb{N}}$ of $\mathbb{R}$ valued random variables given on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is conditionally iid given some sub-σ field $\mathcal{G}$, then the sequence is exchangeable.

6. (*15 points*) Let E be a countable set and let $p = (p(i,j))_{i,j \in E}$ be a transition probability matrix on E. Let $\{X_n\}_{n \in \mathbb{N}_0}$ be the corresponding Markov chain. Suppose that the chain is irreducible and there is a transient state in E. Show that there is no invariant distribution for the chain.