STOR 635 CWE (2022)

Read the following information before starting the exam: Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer. There are 5 problems.

For multipart questions, even if you do not solve a part which involves showing something, you can use it to solve subsequent parts.

Good luck!

1. (10 points) Let $\{X_i\}_{i\in\mathcal{I}}$ be a uniformly integrable family and $\{\mathcal{G}_j\}_{j\in\mathcal{J}}$ be a collection of sub σ-fields of $\mathcal{F}$. Show that the collection $\mathcal{U} \doteq \{E(X_i \mid \mathcal{G}_j),\ (i,j) \in \mathcal{I}\times\mathcal{J}\}$ is a uniformly integrable family.

2. (8 points) Suppose that $\{X_n\}$ is an integrable $\mathcal{F}_n$-adapted sequence such that $E(X_\tau) = E(X_0)$ for every bounded $\mathcal{F}_n$-stopping time τ. Show that $\{X_n\}$ must be a $\mathcal{F}_n$-martingale.

3. (10 points) Let X_n and Y_n be positive integrable and adapted to $\mathcal{F}_n$. Suppose that $\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) \leq X_n + Y_n$ with $\sum Y_n < \infty$ a.s. Prove that X_n converges a.s. to a finite limit.

[Hint: Introduce a supermartingale; for any given $M > 0$, consider the stopping time $\tau = \inf_k \sum_{m=1}^k Y_m > M$ and stop the supermartingale at time τ. Also use the fact about a.s. convergence of non-negative super-martingales.]

4. (12 points) Let $\{Y_i : i \in \mathbb{N}\}$ be independent non-negative random variables with mean one. Let $a_k := \mathbb{E}(\sqrt{Y_k}), k \in \mathbb{N}$. Note that $a_k \in (0,1], k \in \mathbb{N}$, by Jensen's inequality. Let $M_0 = 1$ and

$$M_n := \prod_{i=1}^n Y_i,\ n \in \mathbb{N}.$$

(i) (4 points) Show that $M_n \to M_\infty$ almost surely as $n \to \infty$ for some random variable M_∞ with $\mathbb{E}(M_\infty) \leq 1$, and $\{M_n : n \geq 0\}$ is uniformly integrable if and only if $\mathbb{E}(M_\infty) = 1$.

(ii) (8 points) Show that $\mathbb{E}(M_\infty) = 1$ if and only if $\prod_{i=1}^\infty a_i > 0$. Also show that, if $\prod_{i=1}^\infty a_i = 0$, then $M_\infty = 0$ almost surely.

[Hint: Consider the martingale N with $N_0 = 1$ and $N_n := \prod_{i=1}^n \frac{\sqrt{Y_i}}{a_i}$, $n \in \mathbb{N}$, and note that $M_n \leq N_n^2$ for all n.]

5. (10 points)

(i) (4 points) Consider a Markov chain $\{X_n : n \geq 0\}$ starting at $X_0 \in \{0, 1, \dots\}$ and with transition probabilities $p_{i,i+1} = 1/m$, $p_{i,0} = 1 - \frac{1}{m}$, for $i \in \{0, 1, \dots\}$. Fix $k \in \mathbb{N}$. Let

$$\tau := \inf\{n \geq 0 : X_n = k\}.$$

Let $\Psi(j) := \mathbb{E}_j(\tau), 0 \leq j \leq k-1$, where $\mathbb{E}_j$ denotes expectation corresponding to the starting point $X_0 = j$. Show that

$$\Psi(j) = 1 + \left(1 - \frac{1}{m}\right)\Psi(0) + \frac{1}{m}\Psi(j+1),\ \ 0 \leq j \leq k-1.$$

(ii) (6 points) Assume an experiment has m equally probable outcomes. Suppose trials of the experiment are repeated independently. Show that, for any $k \in \mathbb{N}$, the expected number of trials required to see k consecutive occurrences of a given outcome is $m(m^k - 1)/(m-1)$.