STOR641 - Comprehensive Written Exam - August 2021

This test consists of three questions. The relative weights are specified following each part. You are NOT allowed to refer to any notes, books etc.

Good luck!

1− Consider a retailer shop located on the second floor of a building. Customers arrive at the first floor and take the elevator to reach to the second floor. Once they are on the second floor, they spend some time in the shop, and once they are done, they take the elevator to go down to the first floor and leave the building. To simplify things, assume that the whole building has a capacity of two so that customers who arrive when there are already two in the building go away. Although it is not necessary for modeling, you can also assume that the system starts so that there is noone in the building and the elevator is on the first floor. As soon as a customer arrives at the building or is done with shopping and ready to leave the building, the customer calls the elevator to go up or down unless the elevator is already on the same floor as the customer. The elevator does not go up to the second floor unless there are at least two people who want to go up or someone called the elevator to the second floor to go down (from the second floor down, the elevator would work even if there is a single person), and it responds to the requests in the order they arrive. In particular, if someone calls the elevator when it is on the move, it first completes the trip it started, and then goes to wherever the next request in the call queue came from. Suppose that customers arrive at the first floor according to an independent Poisson process with rate λ $(0 < \lambda < \infty)$. The amount of time customers spend in the shop is a sequence of iid exponentially distributed random variables with rate θ $(0 < \theta < \infty)$ and the times it takes the elevator to go either up from the first floor to the second floor or down from the second floor to the first floor is a sequence of iid exponentially distributed random variables with parameter μ $(0 < \mu < \infty)$.

- **(a)** Model this system as a continuous-time Markov chain. Clearly describe the state space $\mathcal{S}$ and provide the transition rates.**(20 points)**
- **(b)** For any $j \in \mathcal{S}$, let p_j denote the steady-state probability that the system is in state j and suppose that each customer who visits the store leaves an expected revenue of \$$r$. Give an expression for the long-run per unit time revenue for the shop. Give your answer using p_js. DO NOT WRITE OR SOLVE THE BALANCE EQUATIONS. **(10 points)**
- **(c)** For any $j \in \mathcal{S}$, let p_j denote the steady-state probability that the system is in state j. Give an expression for the long-run average (per customer) total time customers who enter the building spend waiting either for the elevator to arrive (both on the way up or on the way down) or another customer to arrive to go up to the second floor. Give your answer using p_js. DO NOT WRITE OR SOLVE THE BALANCE EQUATIONS. **(10 points)**

2− A soccer team will play a series of games. The team gets three points if it is a win and one point if it is a draw. If it loses, it gets no points. As long as the total points the team accumulated over the last three games is greater than or equal to four but less than nine, it will continue playing. If the team reaches nine points over the last three games, then it will qualify for a tournament and stop playing new games. If it cannot accumulate at least four points over the last three games, then it will fail to qualify for the tournament and again stop playing. Suppose that the outcome of each game is independent of others and the team wins with probability $p > 0$, gets a draw with probability $q > 0$, and loses with probability $r > 0$, where $p + q + r = 1$.

(a) Suppose that the team has played three games so far and will continue to play. Write a system of linear equations whose solution would give you the probability that the team will eventually qualify for the tournament given the outcome of the last three games. DO NOT SOLVE THE EQUATIONS. **(10 points)**

(b) Suppose that you solved the equations in part (a) and obtained the desired probabilities. Using your notation in part (a), give an expression for the probability that the team will qualify if it has not played any games yet. **(10 points)**

(c) Suppose that now every time the team fails to qualify, it starts playing from scratch. This continues until the team finally qualifies. Explain how you can determine the expected number of games the team will play. You can either provide the system of linear equations whose solution will give you the desired result or simply provide a general outline for how you would find the expectation. **(10 points)**

3– Suppose that customers arrive at a store according to a Poisson process with rate λ $(0 < \lambda < \infty)$ per minute.

(a) Suppose that Jill and John are two of the customers who visited the store. Jill arrived first, and John was the next customer to arrive. Give an expression for the probability that Jill and John arrived within 15 minutes of each other. **(10 points)**

(b) Suppose that Jill and John both arrived at the store between 1 pm and 2 pm but we do not know in which order they arrived or whether there were other customers who visited the store during the same hour. What is the probability that Jill and John arrived within 15 minutes of each other?**(10 points)**

(c) Suppose again that Jill and John both arrived at the store between 1 pm and 2 pm. We again do not know in which order they arrived but we know that there were eight other customers who visited store during the same hour. What is the probability that Jill and John arrived within 15 minutes of each other? **(10 points)**