# STOR 642
# Comprehensive Written Examination
# 9:00am-1:00pm, August 12, 2021

This test consists of three questions.
This is a closed book exam. No communication devices of any kind are allowed.
Explain your answers in detail.
The problem and subproblem weights are given in parentheses.
Contact me at 919 491 9408 by phone or text if you have any questions.

**Problem 1. Jackson Networks.** (36 points)

Consider a network of $N$ single server queueing stations. Let $\lambda_i$ be the external arrival rate to station $i$, and $\mu_i$ be the service rate of the server at station $i$ $(1 \leq i \leq N)$. Suppose a fraction $r_{i,j}$ of customers that complete service at station $i$ go to station $j$, and the rest leave the network.

1. (6) List all the assumptions needed to model this as a Jackson Network. Assume these assumptions are satisfied in the rest of this problem.
2. (6) Let $a_i$ be the total arrival rate (internal plus external) to node $i$. State the traffic equations satisfied by the $a_i$'s.
3. (6) State the condition of stability for this network.
4. (6) State the limiting distribution of the state of this network assuming it is stable.
5. (6) Compute the expected number of customers in the network in steady state, assuming it is stable.
6. (6) Suppose

$$a = \sum_{i=1}^{N} a_i.$$

Let $\mu > a$ be the total service rate that is available to allocate among the $N$ stations so that

$$\sum_{i=1}^{N} \mu_i = \mu.$$

Show that the optimal allocation that minimizes the expected total number of customers in the network is given by

$$\mu_i = a_i + (\mu - a)\frac{\sqrt{a_i}}{\sum_{j=1}^{N} \sqrt{a_j}}, \quad 1 \leq i \leq N.$$

**Problem 2. Renewal Processes.** (28 Points).

Let $\{N(t), t \geq 0\}$ be a renewal process generated by a sequence of iid non-negative random variables $\{X_n, n \geq 1\}$, with common cdf $F$, mean $\tau < \infty$ and variance $\sigma^2$.

1. (3) State the almost sure version of the elementary renewal theorem. Do not prove it.
2. (3) Define the renewal function $M(t)$.
3. (3) State the elementary renewal theorem. Do not prove it.
4. (4) Derive the renewal equation satisfied by $M(t)$.
5. (3) Let $\tilde{M}(s)$ and $\tilde{F}(s)$ be the LSTs of $M(\cdot)$ and $F(\cdot)$, respectively. Show that

$$\tilde{M}(s) = \frac{\tilde{F}(s)}{1 - \tilde{F}(s)}.$$

6. (6) Let $K > 1$ be a given integer. Define $Z(t) = N(t) \pmod{K}$; that is, $Z(t)$ is the remainder when $N(t)$ is divided by $K$. Show that $\{Z(t), t \geq 0\}$ is a semi-Markov process. What is its state space $S$ and kernel $G(x)$?
7. (6) State the condition under which $\{Z(t), t \geq 0\}$ is positive recurrent and aperiodic. Assume this condition is satisfied and compute

$$\lim_{t \to \infty} P(Z(t) = j), \quad j \in S.$$

**Problem 3. Optimal Dealership Inventory Management.** (36 Points).

A used car dealer manages his car inventory by buying (at low prices) and selling (at high prices) cars and makes money in the process. His car lot is large enough to store $K$ cars at most. Each day the dealer advertises a bid price $p$ and an ask price $q$. Assume that $p, q \in \{r_1, r_2, \cdots, r_M\}$, a set of $M$ prices, listed in increasing order. If he advertises the bid price $p$ and the ask price $q$, the probability that he buys one car that day is $b(p)$ and sells one car is $s(q)$, and the probability that he buys zero cars is $1 - b(p)$ and sells zero cars is $1 - s(q)$. Assume that

$$0 < b(r_1) < b(r_2) < \cdots < b(r_M) < 1$$

and

$$1 > s(r_1) > s(r_2) > \cdots > s(r_M) > 0.$$

The buying and selling events are independent. If he buys a car at price $p$, the cash level goes down by $p$, and if he sells a car at price $q$, the cash level goes up by $q$. Let $X_n$ be the number of cars on the parking lot on day $n$. If $X_n = 0$, he is not allowed to sell any cars, and if $X_n = K$, he is not allowed to buy any cars on that day. The cash level can be positive or negative. The dealer wants to set his bid and ask prices as a function of the inventory so as to generate the maximum average cash flow per day in the long run.

1. (6) Formulate this as an MDP. State the state space, action space, transition probabilities, and one step rewards.

2. (6) Write the optimality equation for $g$, the optimal long run average reward per day, and $h$, the bias function.

3. (6) Show that the MDP is unichain, and hence the solution exists.

4. (6) Show how to use value iteration to find the optimal policy.

5. (6) Show how to use policy iteration to find the optimal policy.

6. (6) Show how to use linear programming (either the primal or the dual) to find the optimal policy.