1. Let $X_1, X_2, \ldots$ be i.i.d. sample from a Pareto($\lambda, \gamma$), i.e.,

$$f(x|\lambda, \gamma) = \frac{\gamma \lambda}{(1 + \lambda x)^{\gamma + 1}} I_{(0, \infty)}(x), \quad \lambda > 0, \gamma > 0.$$ 

In some of the problem parts below you might need to make extra assumptions? If you make extra assumptions, state them clearly.

(a) Does this two parameter Pareto($\lambda, \gamma$) form an exponential family?
(b) Find the Method of Moments (MM) estimator $\hat{\theta}_{MM}$ based on first two moments.
(c) Is $\hat{\theta}_{MM}$ strongly consistent?
(d) Is $\hat{\theta}_{MM}$ asymptotically normal? If yes, what is its asymptotic variance?
(e) Is $\hat{\theta}_{MM}$ asymptotically efficient? If it is not, how would you improve it by scoring?

For the rest of this problem we assume $\lambda = 1, \gamma > 2$ and consider the U-statistics:

$$U_n = \left( \begin{array}{c} n \\ 2 \end{array} \right)^{-1} \sum_{1 \leq i < j \leq n} (X_i(1 + X_j) \log(1 + X_j) + X_j(1 + X_i) \log(1 + X_i))$$

(f) Find $\eta = EU_n$.
(g) Find Hájek projection $\pi_n(U_n - \eta)$.
(h) Does $\sqrt{n}(U_n - \eta) \xrightarrow{D} N(0, \xi^2)$? If yes, find $\xi^2$.

Hint: The following integrals might be helpful throughout:

For $\gamma > 1$

$$\int_0^\infty \frac{x \gamma}{(1 + x)^{\gamma + 1}} dx = \frac{1}{\gamma - 1}, \quad \int_0^\infty \frac{(1 + x) \log(1 + x) \gamma}{(1 + x)^{\gamma + 1}} dx = \frac{\gamma}{(\gamma - 1)^2}$$
For $\gamma > 2$

$$\int_0^\infty \frac{x^2 \gamma}{(1 + x)^{\gamma+1}} \, dx = \frac{2}{(\gamma - 1)(\gamma - 2)}$$

2. Let $X_1, \ldots, X_n$ be i.i.d. sample from Negative Binomial(2, $p$) distribution, i.e.

$$f(x|p) = (x + 1)p^2(1 - p)^x I_{\{0,1,\ldots\}}(x).$$

Consider the $N(0, 1)$ prior on the log odds ratio $\psi = \log(p/(1 - p))$. What is a suitable Gaussian approximation for the posterior $p(\psi|x_1, \ldots, x_n)$ for large $n$? In what sense is this approximation asymptotically valid?