Statistics 655 Comprehensive Written Exam
August 2022

All problems have equal weight; partial credit will be given for each part of a problem. In budgeting your time, note that some parts will take longer than others. In many cases, different parts of a problem can be done independently, so if you are unsure how to handle one part of a problem, don't hesitate to try the others. No answer should require a great deal of computation or a complicated proof.

1. Let $Y = \mu + Z \in \mathbb{R}^d$ be a random vector where $\mu \in \mathbb{R}^d$ is a fixed but unknown vector and $Z \sim \mathcal{N}_d(0, I)$.

   (a) What is the distribution of $||Z||^2$?

   (b) Find $\mathbb{E}||Z||^2$ and $\text{Var}(||Z||^2)$.

   (c) What is the distribution of $\langle \mu, Z \rangle$?

Now suppose that we observe Y and wish to estimate $||\mu||^2$. Consider the estimator U^2 defined by $U^2 := ||Y||^2 - \mathbb{E}||Z||^2$.

   (d) Is U^2 unbiased? Justify your answer.

   (e) Find an upper bound on $\mathbb{E}|U^2 - ||\mu||^2|$ in terms of $\sqrt{p}$ and $||\mu||$.

2. Complete the following.

   (a) Let $X \in \{0, 1, 2, \ldots\}$ be a random variable taking values in the non-negative integers. Show that

$$\mathbb{P}(X \geq 1) \geq \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}$$

   (b) Let $X_1, X_2, \ldots \in \{0, 1, 2, \ldots\}$ be a random variables such that for some positive constant c the inequality $\text{Var}(X_n) \leq c\,\mathbb{E}X_n$ holds for each n, and $\mathbb{E}X_n \to \infty$ as n tends to infinity. Show that $\mathbb{P}(X_n = 0) \to 0$ as n tends to infinity.

3. Let $U_1, U_2, \ldots, U \in \mathbb{R}$ be random variables and let $X_1, X_2, \ldots, X \in \mathbb{R}^d$ be random vectors defined on the same probability space, such that $U_n \to U$ in probability and $X_n \Rightarrow X$ in law.

(a) Does $X_n\, X \Rightarrow X^2$? Prove this or give a counterexample.

(b) Let $Y_n = X_n\, e^{-X_n^2}$. What can you say about the behavior of Y_n and $\mathbb{E}Y_n$ as n tends to infinity? Justify your answer.

(c) Let $h : \mathbb{R} \to \mathbb{R}$ be bounded and uniformly continuous. Does $\mathbb{E}h(U_n X_n) - \mathbb{E}h(U X_n) \to 0$ as n tends to infinity? Justify your answer.

(d) Argue that $U_n\, X_n \Rightarrow UX$ if U is independent of $X_1, X_2, \ldots, X$.

4. Let $X_1, X_2, \ldots, X \in \mathbb{R}$ be iid random variables with $\mathbb{E}|X|^4 < \infty$, mean μ, and variance σ^2.

(a) Argue carefully that $\mathbb{E}|X|^4 < \infty$ ensures that μ is well defined and finite.

(b) Write down the the sample variance S_n^2 of $X_1, \ldots, X_n$.

(c) Let $\tilde{S}_n^2 = n^{-1} \sum_{i=1}^{n} (X_i - \mu)^2$. Is there an inequality relating $\mathbb{E}[\tilde{S}_n^2/(1+\tilde{S}_n^2)]$ and $\sigma^2/(1+\sigma^2)$?

(d) Establish a relationship of the form $S_n^2 = \tilde{S}_n^2 + C_n$ where C_n can be expressed in terms of stochastic order symbols and the sample size n.

(e) Find the asymptotic distribution of S_n^2 in terms of the central moments $c_k = \mathbb{E}(X - \mu)^k$.

(f) What can you say about the asymptotic distribution of $S_n^2/(1 + S_n^2)$? Justify your answer.