Consider a regression with two predictors \( x_{i1}, x_{i2}, i = 1, \ldots, n \), and assume the model

\[
y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i1} x_{i2} + \epsilon_i, \quad i = 1, \ldots, n
\]

where \( \beta_0, \ldots, \beta_3 \) are unknown parameters and \( \epsilon_i \sim N(0, \sigma^2) \) are independent errors with common unknown variance \( \sigma^2 \). Note that the model has just the interaction term \( \beta_3 x_{i1} x_{i2} \) but no terms in \( x_{i1}^2 \) or \( x_{i2}^2 \). It is natural to want to test whether \( H_0 : \beta_3 = 0 \).

Defining \( S_{jk} = \sum_{i=1}^{n} x_{i1}^j x_{i2}^k \) let us further assume: \( S_{10} = S_{01} = S_{11} = S_{12} = 0 \) but that \( S_{20}, S_{02}, S_{21}, S_{22} \) are not 0.

(a) Find explicit expressions for the least squares estimators \( \hat{\beta}_0, \ldots, \hat{\beta}_3 \) in terms of the \( S_{jk} \)'s and \( \sum y_i, \sum y_i x_{i1}, \sum y_i x_{i2} \) and \( \sum y_i x_{i1} x_{i2} \). [12 points]

(b) Find expressions for the standard errors of \( \hat{\beta}_0, \ldots, \hat{\beta}_3 \), in terms of \( n, S_{20}, S_{02}, S_{21}, S_{22} \) and the residual standard deviation \( s \) (assuming that \( s^2 \) is the standard unbiased estimator of \( \sigma^2 \)). [5 points]

(c) A \( t \)-test of significance level \( \alpha \) will reject \( H_0 \) if \( |\frac{\hat{\beta}_3}{s}| > C \) for some \( C \) which is a combination of \( n, S_{20}, S_{02}, S_{21}, S_{22} \) and \( \alpha \) (alpha). Find \( C \). (You may, if you wish, express your answer as an appropriate R function.) [5 points]

(d) What is the power of the test in (c) when \( \beta_3 \neq 0 \)? You answer should be expressed in terms of the given parameters and relevant percentage points of the noncentral \( t \) or \( F \) distributions. (You may choose to express your answer as an R function though alternatives are also acceptable if the derivation behind your answer is clearly explained.) (Hint: First find the distribution of \( \frac{\hat{\beta}_3^2}{s^2} \) when \( \beta_3 \neq 0 \).) [16 points]

(e) Show that the observation with highest leverage is the index \( i \) that maximizes

\[
x_{i2}^2 (S_{20} S_{22} - S_{21}^2) + x_{i1}^2 S_{02} (S_{22} - 2 S_{21} x_{i2} + x_{i2}^2 S_{20}).
\]

[12 points]