Optimization Qualifying Examination

Information:
- Student’s full name: ________________________________
- Student’s signature: _______________________________
- Date: August 14, 2023.
- Time: 9:00 AM - 1:00 PM.
- Honor pledge: I have neither given nor received unauthorized assistance on this exam.

General instructions: Please carefully read the following instructions before answering.

a. Write your name on this sheet. Date and sign on the first page.
b. This examination is closed-books and notes, and consists of two questions.
c. Answer all two as clearly and concisely as you are able.
d. Use of the internet, computers, and/or mobile devices is not permitted.

The exam questions

Question 1: (50 points) Consider the following convex quadratic program:

\[
\min_{x \in \mathbb{R}^p} \left\{ f(x) := \frac{1}{2} x^T Q x + q^T x \quad \text{subject to} \quad Ax = b \right\},
\]

where \( Q \) is a \( p \times p \) symmetric positive semidefinite matrix, \( q \in \mathbb{R}^p \), \( A \in \mathbb{R}^{n \times p} \) is a full-row rank matrix such that \( \text{rank}(A) = n < p \), and \( b \in \mathbb{R}^n \). Consider the dual function of (QP) defined as

\[
d(y) := \min_{x \in \mathbb{R}^p} \left\{ \frac{1}{2} x^T Q x + q^T x + y^T (Ax - b) \right\}.
\]

Solve the following questions:

(a) Prove that \( d \) is a concave quadratic function (i.e., \( -d \) is a convex quadratic function).

(b) Prove that if \( d \) is strictly concave, then it is also strongly concave.
(c) Explicitly solve \((QP)\) when \(Q = I\), the identity matrix.

Note: A quadratic function is a function of the form \(d(y) = \frac{1}{2} y^T H y + h^T y\) for a given symmetric matrix \(H\) and a vector \(h\).

**Question 2:** (50 points) We call a sequence of zeros and ones a *string*. We say that a string \(w\) is a *superstring* of the string \(x\) if \(w = sxt\) where \(s\) and \(t\) are strings.

We say that a string \(w\) is a *common superstring* of the strings \(x\) and \(y\) if \(w\) is a superstring of both \(x\) and \(y\). For example, if \(x = 101\) and \(y = 11\) then 10111 is trivially a common superstring of \(x\) and \(y\). But 1011 is also a common superstring of \(x\) and \(y\).

Given the strings \(x_1, \ldots, x_m\) we want to find the shortest common superstring of \(x_1, \ldots, x_m\).

Formulate as an IP. The number of variables and constraints in your formulation should be polynomial in \(m\) and \(\ell_1 + \ldots + \ell_m\) where \(\ell_i\) is the length of \(x_i\) for all \(i\).

Clearly define variables and explain the meaning of all constraints.