Department of Statistics and Operations Research
The University of North Carolina at Chapel Hill
——————————— August, 2023 ———————————

Optimization Qualifying Examination

Information:

- **Student's full name:** ____
- **Student's signature:** __________________________
- **Date:** August 14, 2023. **Time:** 9:00 AM - 1:00 PM.
- **Honor pledge:** I have neither given nor received unauthorized assistance on this exam.

General instructions: Please carefully read the following instructions before answering.

a. Write your name on this sheet. Date and sign on the first page.
b. This examination is closed-books and notes, and consists of two questions.
c. Answer all two as clearly and concisely as you are able.
d. Use of the internet, computers, and/or mobile devices is not permitted.

The exam questions

Question 1: (50 points) Consider the following convex quadratic program:

$$\min_{x\in\mathbb{R}^p}\left\{f(x) := \frac{1}{2}x^TQx + q^Tx \quad \text{subject to} \quad Ax = b\right\}, \tag{QP}$$

where Q is a $p \times p$ symmetric positive semidefinite matrix, $q \in \mathbb{R}^p$, $A \in \mathbb{R}^{n\times p}$ is a full-row rank matrix such that $\operatorname{rank}(A) = n < p$, and $b \in \mathbb{R}^n$. Consider the dual function of (QP) defined as

$$d(y) := \min_{x\in\mathbb{R}^p}\left\{\frac{1}{2}x^TQx + q^Tx + y^T(Ax - b)\right\}.$$

Solve the following questions:

(a) Prove that d is a concave quadratic function (i.e., $-d$ is a convex quadratic function).

(b) Prove that if d is strictly concave, then it is also strongly concave.

(c) Explicitly solve (QP) when $Q = \mathbb{I}$, the identity matrix.

Note: A quadratic function is a function of the form $d(y) = \frac{1}{2}y^T Hy + h^T y$ for a given symmetric matrix H and a vector h.

Question 2: (50 points) We call a sequence of zeros and ones a *string*. We say that a string w is a *superstring* of the string x if $w = sxt$ where s and t are strings.

We say that a string w is a *common superstring* of the strings x and y if w is a superstring of both x and y. For example, if $x = 101$ and $y = 11$ then 10111 is trivially a common superstring of x and y. But 1011 is also a common superstring of x and y.

Given the strings $x_1, \dots, x_m$ we want to find the shortest common superstring of $x_1, \dots, x_m$.

Formulate as an IP. The number of variables and constraints in your formulation should be polynomial in m and $\ell_1 + \dots \ell_m$ where ℓ_i is the length of x_i for all i.

Clearly define variables and explain the meaning of all constraints.

The end