STOR 634, CWE 2022-23.

Each problem is 10 points. There are 5 problems in all.

1. (*10 points*)

a. (*2 pts*) Suppose that $\{X_\alpha\}_{\alpha\in\mathcal{I}}$ is a family of real valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Say what it means for this family to be uniformly integrable.

b. (*3 pts*) Suppose that the above family is uniformly integrable. Show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $\mathbb{P}(A) < \delta$,

$$\sup_{\alpha\in\mathcal{I}} \int_A |X_\alpha| d\mathbb{P} < \varepsilon.$$

c. (*2 pts*) Suppose now that $\mathcal{I} = \mathbb{N}$ so that $\{X_n\}$ is a sequence of uniformly integrable random variables. Suppose that $X_n \to X$ in probability for some real random variable X. Show that $\mathbb{E}|X| < \infty$ and that the sequence $\{|X_n - X|\}$ is uniformly integrable.

d. (*3 pts*) With $\{X_n\}$ and X as in part (c) show that $X_n \to X$ in $\mathcal{L}^1$. [Hint: Consider the set $\{|X_n - X| \le M\}$ and its complement.]

2. (*10 points*)

a. (*4 pts*) State Kolmogorov's three series theorem.

b. (*6 pts*) Let $\{X_n\}$ be a sequence of independent random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\psi(x) = x^2$ when $|x| \le 1$ and $\psi(x) = |x|$ when $|x| \ge 1$. Suppose that $\mathbb{E}X_n = 0$ for every n and $\sum_{n=1}^\infty \mathbb{E}\psi(X_n) < \infty$. Show that $\sum_{n=1}^\infty X_n$ converges a.s.

3. (*10 points*) Let $\{\mu_n\}$ be a sequence of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and let μ be another probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

a. (*2 pts*) Say what it means for the sequence $\{\mu_n\}$ to be tight.

b. (*4 pts*) Suppose that for every continuous real valued function f on $\mathbb{R}$ with compact support $\int f d\mu_n \to \int f d\mu$. Show that the sequence $\{\mu_n\}$ is tight. [Hint: For $M < \infty$ consider a continuous nonnegative function f that is 1 on $[-M, M]$ and zero outside $[-(M+1), M+1]$]

c. (*4 pts*) With $\{\mu_n\}$ as in part (b), show that μ_n converges weakly to μ. [Hint: Use tightness and approximate the indicator function of an interval $[a, b]$ by continuous nonnegative functions of compact support in a suitable manner.]

4. (*10 points*) Let μ_n be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with ch.f. φ_n.

a. (*2 pts*) Say what it means for the sequence $\{\varphi_n\}$ to be equicontinuous.

b. (*4 pts*) Suppose that $\{\mu_n\}$ is tight. Show that the sequence $\{\varphi_n\}$ is equicontinuous.

c. (*4 pts*) Suppose next that $\mu_n \to^d \mu_\infty$. Show that $\varphi_n(t) \to \varphi_\infty(t)$ uniformly in $t \in [0, T]$ for every T, where φ_∞ is the ch.f. of μ_∞.

5. (*10 points*) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{A_n\}$ be a sequence of events in $\mathcal{F}$.

a. (*3 pts*) Give the definition of $\limsup A_n$ and $\liminf A_n$. Show that $\liminf A_n \subset \limsup A_n$.

b. (*3 pts*) Show that $1_{\limsup A_n} = \limsup 1_{A_n}$.

c. (*4 pts*) Let $\{X)_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}$-valued random variables on the above probability space. Suppose that $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \varepsilon) < \infty$. Show that $\{\lim_{n\to\infty} X_n \text{ exists and equals } 0\}$ is in $\mathcal{F}$ and that $\mathbb{P}(\lim_{n\to\infty} X_n = 0) = 1$.