STOR 635 Exam: CWE Year: 2023

There are 4 questions. Attempt all questions. You may appeal to any result proved in class without proof unless you are specifically asked to "**Give a complete proof.** ". State any result you use. All questions are worth the same number of total points (10 points). Points for parts of a question can be found in boxes on the right. *Even if you don't know the complete solution DONT STRESS OUT. Put down your attempt.* Partial credit will be assigned so try your best on each question.

1. $\boxed{\text{10 points}}$ Recall that we talked about conditional expectations in class given a sub-sigma field $\mathcal{G} \subseteq \mathcal{F}$. For any event $A \in \mathcal{F}$, define

$$\mathbb{P}(A|\mathcal{G}) := \mathbb{E}(\mathbb{1}\{A\}|\mathcal{G})$$

namely the conditional expectation of the indicator function of the set A given $\mathcal{G}$. Now for $i = 1, 2$, let X_i be random variables defined on $(\Omega, \mathcal{F}, P)$ taking values in space S_i (with associated sigma-fields $\mathcal{S}_i$). Let $\mathcal{G}$ be a sub-sigma field of $\mathcal{F}$. Prove that the following assertions (a), (b), (c) are **equivalent**. When any one of these assertions hold, we says X_1, X_2 are <u>conditionally independent given $\mathcal{G}$</u>.

(i) $\mathbb{P}(X_1 \in A_1, X_2 \in A_2|\mathcal{G}) = \mathbb{P}(X_1 \in A_1|\mathcal{G})\,\mathbb{P}(X_2 \in A_2|\mathcal{G})$ for all $A_i \in \mathcal{S}_i$.

(ii) $\mathbb{E}(h_1(X_1)h_2(X_2)|\mathcal{G}) = \mathbb{E}(h_1(X_1)|\mathcal{G})\,\mathbb{E}(h_2(X_2)|\mathcal{G})$ for all bounded measurable $h_i : \mathcal{S}_i \to \mathbb{R}$.

(iii) $\mathbb{E}(h_1(X_1)|\mathcal{G}, X_2) := \mathbb{E}(h_1(X_1)|\mathcal{G})$ for all bounded measurable $h_1 : S_1 \to \mathbb{R}$. Here when we write $\mathbb{E}(\cdot|\mathcal{G}, X_2)$ for the conditional expectation $\mathbb{E}(\cdot|\mathcal{G}^*)$ where

$$\mathcal{G}^* = \sigma(\mathcal{G} \cup \sigma(X_2))$$

Note: A π class generating this sigma field is

$$\mathcal{P} := \left\{X_2^{-1}(B) \cap A : B \in \mathcal{S}_2, A \in \mathcal{G}\right\}$$

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2. $\boxed{\text{10 points}}$ Let S_n be the total assets of an insurance company at the end of year n. Suppose that in year n the company receives premiums of c and pays claims totaling ξ_n, where ξ_n are independent with Normal(μ, σ^2) distribution, where $0 < \mu < c$. The company is ruined if its assets fall to zero or below. Show

$$\mathbb{P}(\text{ruin}) \leqslant \exp(-2(c-\mu)S_0/\sigma^2).$$

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3. $\boxed{\text{10 points}}$ Suppose $\{X_n\}$ and $\{Y_n\}$ be two independent Markov chains on a countable state space S (without loss of generality assume $S = \{1, 2, \ldots,\}$) with $X_0 = i$ and $Y_0 = j$ with the same transition matrix $\mathbf{P} = (P(x,y)_{x,y \in S}$. Then (you do not need to assume this) $\{(X_n, Y_n) : n \geqslant 0\}$ is is a Markov chain on state space $S \times S$. Assume the transition matrix $\mathbf{P}$ results in the corresponding Markov chain being irreducible and positive recurrent.

(a) $\boxed{\text{7 points}}$ If $\mathbf{P}$ is aperiodic, show that this implies the product chain is also irreducible and positive recurrent.

(b) $\boxed{\text{3 points}}$ Give a counter example to show that is the chain is not aperiodic, then the product chain need not be irreducible on the state space $S \times S$.

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4. $\boxed{\text{10 points}}$ Let $\{\mathcal{F}_n : n \geqslant 0\}$ be a filtration and let $\{X_n : n \geqslant 0\}$ be an adapted process with $\mathbb{E}(|X_n|) < \infty$ for all $n \geqslant 0$. Show that $\{X_n : n \geqslant 0\}$ is a martingale if and only if $\mathbb{E}(X_T) = \mathbb{E}(X_0)$ for every bounded stopping time T.

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5. $\boxed{\text{10 points}}$ Suppose $\{X_i : i \geqslant 1\}$ are a sequence of real valued **exchangeable** (not necessarily integrable i.e. one could have $\mathbb{E}(|X_1|) = \infty$) random variables. Further suppose $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is a bounded measurable function. Give a **complete proof** to show that there exists a random variable Y_∞ such that

$$\frac{\sum_{i=1}^n \varphi(X_i)}{n} \xrightarrow{a.s.} Y_\infty$$

If you are planning to use De-Finetti's theorem then give a complete proof to show how one can derive the result above from De-Finetti's theorem. Just writing down De-Finetti implies "conditionally *i.i.d.* and thus we get the above result " will get you at most 3 points unless you can clearly justify how you can use De-Finetti to prove the above result. Note: you can also prove the above result directly without appealing to De-Finetti.

$\boxed{\text{STOR 635 End}}$