STOR 641
Comprehensive Written Exam
August 2023

This test consists of 3 questions.
This is a closed books/notes/references exam.
Explain your answers in detail to receive partial/full credit.
Good luck!

Problem 1 (30 points). A system consists of two components in series (components A and B), and whenever either component fails, a system failure occurs. Each component fails independently after an exponentially distributed amount of time with possibly distinct rates λ_A and λ_B. When a component fails, it is replaced with a brand new one instantaneously.

(a) **(5 points)** What is the expected number of component replacements by time $t \geq 0$?

(b) **(5 points)** What is the probability that a system failure is caused by the failure of component B?

(c) **(5 points)** What is the distribution of the time component B is replaced for the nth time?

(d) **(5 points)** Given that the first component replaced was component A, what is the expected time of the first system failure?

(e) **(5 points)** Given that the first component replaced was component A, what is the expected time of the first failure of component B?

(f) **(5 points)** Given that the first component replaced was component A and the second component replaced was component B, what is the expected time of the first failure of component B?

Problem 2 (37 points). A shuttle bus travels between the three terminals of an airport. From Terminal 1, it is equally likely to go to Terminal 2 or Terminal 3. From Terminal 2, it goes to Terminal 1 with probability 1/3 and to Terminal 3 with probability 2/3. From Terminal 3, the shuttle always goes to Terminal 2. The travel time for a trip from Terminal i to Terminal j is an exponential random variable with mean m_{ij}, where $i, j \in \{1, 2, 3\}$ and $i \neq j$. We have $m_{12} = m_{21} = 4$ minutes and $m_{13} = m_{31} = m_{23} = m_{32} = 6$ minutes. Assume that the time shuttle spends at the terminals are negligible.

(a) **(15 points)** What is the long-run fraction of trips that headed towards Terminal 3? (Define a suitable stochastic process to answer this question.)

(b) **(15 points)** What is the long-run fraction of time the shuttle is headed for Terminal 3? (Define a suitable stochastic process to answer this question.)

(c) **(7 points)** Suppose that the expected number of passengers transported from Terminal i to Terminal j in a single trip is f_{ij}, where $i, j \in \{1, 2, 3\}$ and $i \neq j$. What is the long-run average number of passengers on the shuttle? (Use the stochastic processes defined in parts **(a)** and/or **(b)** to answer this question.)

Problem 3 (33 points). The intensive care unit (ICU) of a small hospital has a capacity of $B \geq 2$ beds and is attended by a single nurse. The demand for the ICU is so high that all beds are occupied at all times. Specifically, a new patient is admitted to the ICU as soon as a patient is discharged from it. The nurse provides two types of service for each patient: general nursing or discharge processing (after which the patient leaves the unit). Each patient's requests follow a Poisson process with rate λ and the probability that a request is a discharge is p, where $0 < p < 1$. The patient requests and the types of requests are independent for all patients. The amount of time it takes for the nurse to complete a nursing request is exponentially distributed with rate μ and a discharge request takes an exponentially distributed amount of time with rate θ. The performance measure of interest is the long-run average number of patients waiting for the nurse's attention.

- **(a)** **(15 points)** Suppose that all requests (nursing or discharge) are processed by the nurse according to a first-come-first-served order. Model this system as a CTMC. Define the state and provide the state space and the transition rates.

- **(b)** **(9 points)** Suppose now that a discharge request has a preemptive priority over nursing requests. In other words, if the nurse is busy with a patient with a nursing request at the time of a discharge request, then the nurse immediately switches attention to the patient with a discharge request. Redo part **(a)** but do not provide transition rates.

- **(c)** **(9 points)** Suppose now that discharge requests have non-preemptive priority over nursing requests. Specifically, the nurse never interrupts a service that has started but whenever there are two types of requests waiting at the time the nurse becomes available, serving a patient with a discharge request is preferred. Redo part **(a)** but do not provide transition rates.