Problem 1 (30 points). A system consists of two components in series (components A and B), and whenever either component fails, a system failure occurs. Each component fails independently after an exponentially distributed amount of time with possibly distinct rates $\lambda_A$ and $\lambda_B$. When a component fails, it is replaced with a brand new one instantaneously.

(a) (5 points) What is the expected number of component replacements by time $t \geq 0$?

(b) (5 points) What is the probability that a system failure is caused by the failure of component B?

(c) (5 points) What is the distribution of the time component B is replaced for the $n$th time?

(d) (5 points) Given that the first component replaced was component A, what is the expected time of the first system failure?

(e) (5 points) Given that the first component replaced was component A, what is the expected time of the first failure of component B?

(f) (5 points) Given that the first component replaced was component A and the second component replaced was component B, what is the expected time of the first failure of component B?

Problem 2 (37 points). A shuttle bus travels between the three terminals of an airport. From Terminal 1, it is equally likely to go to Terminal 2 or Terminal 3. From Terminal 2, it goes to Terminal 1 with probability 1/3 and to Terminal 3 with probability 2/3. From Terminal 3, the shuttle always goes to Terminal 2. The travel time for a trip from Terminal $i$ to Terminal $j$ is an exponential random variable with mean $m_{ij}$, where $i, j \in \{1, 2, 3\}$ and $i \neq j$. We have $m_{12} = m_{21} = 4$ minutes and $m_{13} = m_{31} = m_{23} = m_{32} = 6$ minutes. Assume that the time shuttle spends at the terminals are negligible.

(a) (15 points) What is the long-run fraction of trips that headed towards Terminal 3? (Define a suitable stochastic process to answer this question.)

(b) (15 points) What is the long-run fraction of time the shuttle is headed for Terminal 3? (Define a suitable stochastic process to answer this question.)

(c) (7 points) Suppose that the expected number of passengers transported from Terminal $i$ to Terminal $j$ in a single trip is $f_{ij}$, where $i, j \in \{1, 2, 3\}$ and $i \neq j$. What is the long-run average number of passengers on the shuttle? (Use the stochastic processes defined in parts (a) and/or (b) to answer this question.)
Problem 3 (33 points). The intensive care unit (ICU) of a small hospital has a capacity of \( B \geq 2 \) beds and is attended by a single nurse. The demand for the ICU is so high that all beds are occupied at all times. Specifically, a new patient is admitted to the ICU as soon as a patient is discharged from it. The nurse provides two types of service for each patient: general nursing or discharge processing (after which the patient leaves the unit). Each patient’s requests follow a Poisson process with rate \( \lambda \) and the probability that a request is a discharge is \( p \), where \( 0 < p < 1 \). The patient requests and the types of requests are independent for all patients. The amount of time it takes for the nurse to complete a nursing request is exponentially distributed with rate \( \mu \) and a discharge request takes an exponentially distributed amount of time with rate \( \theta \). The performance measure of interest is the long-run average number of patients waiting for the nurse’s attention.

(a) (15 points) Suppose that all requests (nursing or discharge) are processed by the nurse according to a first-come-first-served order. Model this system as a CTMC. Define the state and provide the state space and the transition rates.

(b) (9 points) Suppose now that a discharge request has a preemptive priority over nursing requests. In other words, if the nurse is busy with a patient with a nursing request at the time of a discharge request, then the nurse immediately switches attention to the patient with a discharge request. Redo part (a) but do not provide transition rates.

(c) (9 points) Suppose now that discharge requests have non-preemptive priority over nursing requests. Specifically, the nurse never interrupts a service that has started but whenever there are two types of requests waiting at the time the nurse becomes available, serving a patient with a discharge request is preferred. Redo part (a) but do not provide transition rates.